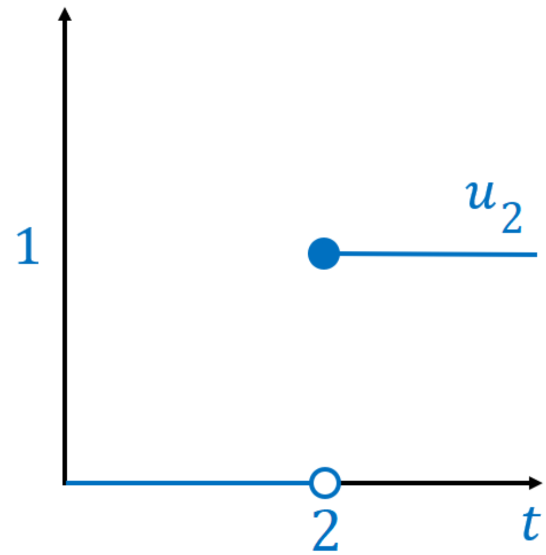
LECTURE: STEP FUNCTIONS (I)

1. Simple Jumps

Definition:

$$u_2(t) = \begin{cases} 0 & \text{if } \ t < 2 \\ 1 & \text{if } \ t \geq 2 \end{cases}$$



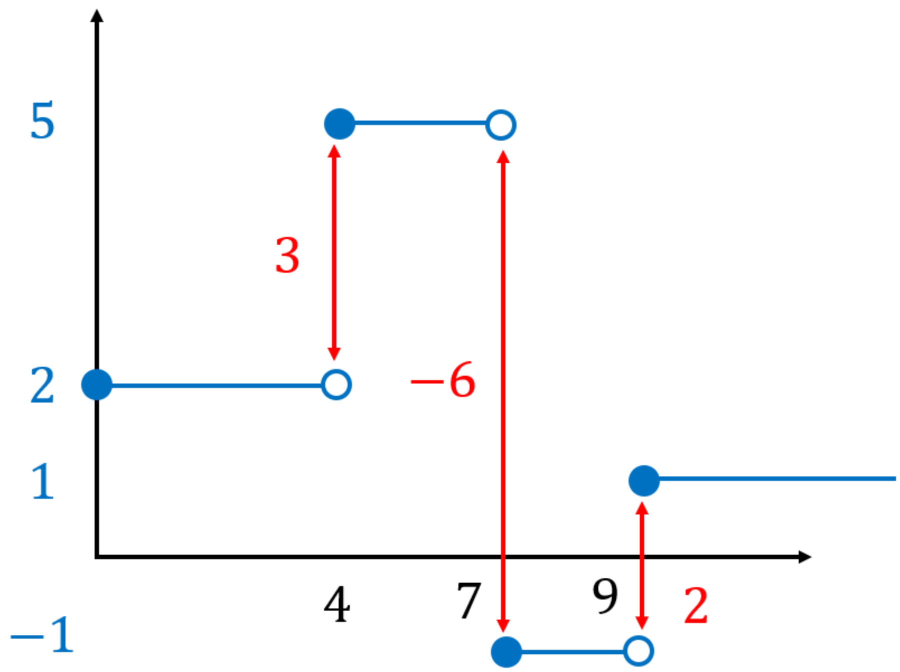
Fact:

$$\mathcal{L}\left\{u_2(t)\right\} = \frac{e^{-2s}}{s}$$

We can use u_c to describe functions that have jumps.

Example 1:

$$\text{Find } \mathcal{L}\left\{f(t)\right\} \text{ where } f(t) = \begin{cases} 2 & \text{if } 0 \leq t < 4 \\ 5 & \text{if } 4 \leq t < 7 \\ -1 & \text{if } 7 \leq t < 9 \\ 1 & \text{if } t \geq 9 \end{cases}$$



STEP 1: Write f in terms of u_c

Start with 2 (initial value)

At $t = 4$, you jump by $3 \Rightarrow +3u_4$

At $t = 7$, you jump by $-6 \Rightarrow -6u_7$

At $t = 9$, you jump by $1 - (-1) = 2 \Rightarrow +2u_9$

$$f(t) = 2 + 3u_4 - 6u_7 + 2u_9$$

STEP 2: Take Laplace Transforms

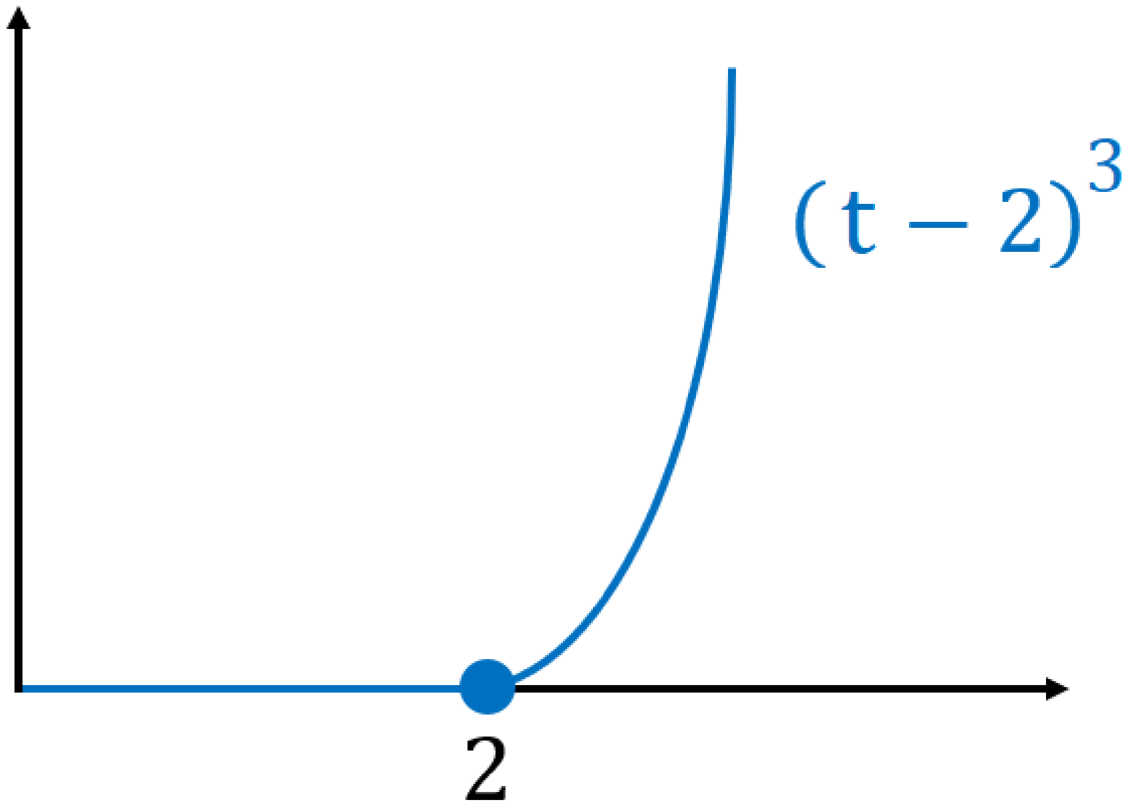
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + 3\mathcal{L}\{u_4\} - 6\mathcal{L}\{u_7\} + 2\mathcal{L}\{u_9\} = \left(\frac{2}{s}\right) + 3\left(\frac{e^{-4s}}{s}\right) - 6\left(\frac{e^{-7s}}{s}\right) + 2\left(\frac{e^{-9s}}{s}\right)$$

2. More general jumps

We can even jump by an entire functions, and not just by constants

Example 2:

$$\text{Find } \mathcal{L}\{g(t)\} \text{ where } g(t) = \begin{cases} 0 & \text{if } t < 2 \\ (t-2)^3 & \text{if } t \geq 2 \end{cases}$$



STEP 1: Write g in terms of u_c

Start at 0 and jump by $(t-2)^3$ at $t=2$ and so

$$g(t) = (t-2)^3 u_2(t)$$

Note: This makes sense because

$$\text{If } t < 2 \text{ then } g(t) = (t-2)^3 \underbrace{u_2(t)}_{0} = 0$$

$$\text{If } t \geq 2 \text{ then } g(t) = (t-2)^3 \underbrace{u_2(t)}_{1} = (t-2)^3$$

STEP 2: Take Laplace Transforms

$$g(t) = (t-2)^3 u_2(t)$$

Slow way:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty (t-2)^3\, u_2(t) e^{-st} dt \\
&= \int_0^2 (t-2)^3\, 0 e^{-st} dt + \int_2^\infty (t-2)^3\, 1 e^{-st} dt \\
&= \int_2^\infty (t-2)^3\, e^{-st} dt \\
&= \int_0^\infty u^3 e^{-s(u+2)} du \qquad (u = t-2) \\
&= \int_0^\infty u^3 e^{-su} e^{-2s} du \\
&= e^{-2s} \int_0^\infty u^3 e^{-su} du \\
&= e^{-2s} \mathcal{L}\left\{t^3\right\} \\
&= e^{-2s} \left(\frac{3!}{s^4}\right) \qquad \text{Because } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \\
&= e^{-2s} \left(\frac{6}{s^4}\right)
\end{aligned}$$

Faster Way:

Fact:

$$\mathcal{L}\{f(t-c)u_c(t)\} = e^{-cs} \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\left\{(t-2)^3\, u_2(t)\right\} = e^{-2s}\mathcal{L}\left\{t^3\right\} = e^{-2s}\left(\frac{6}{s^4}\right)$$

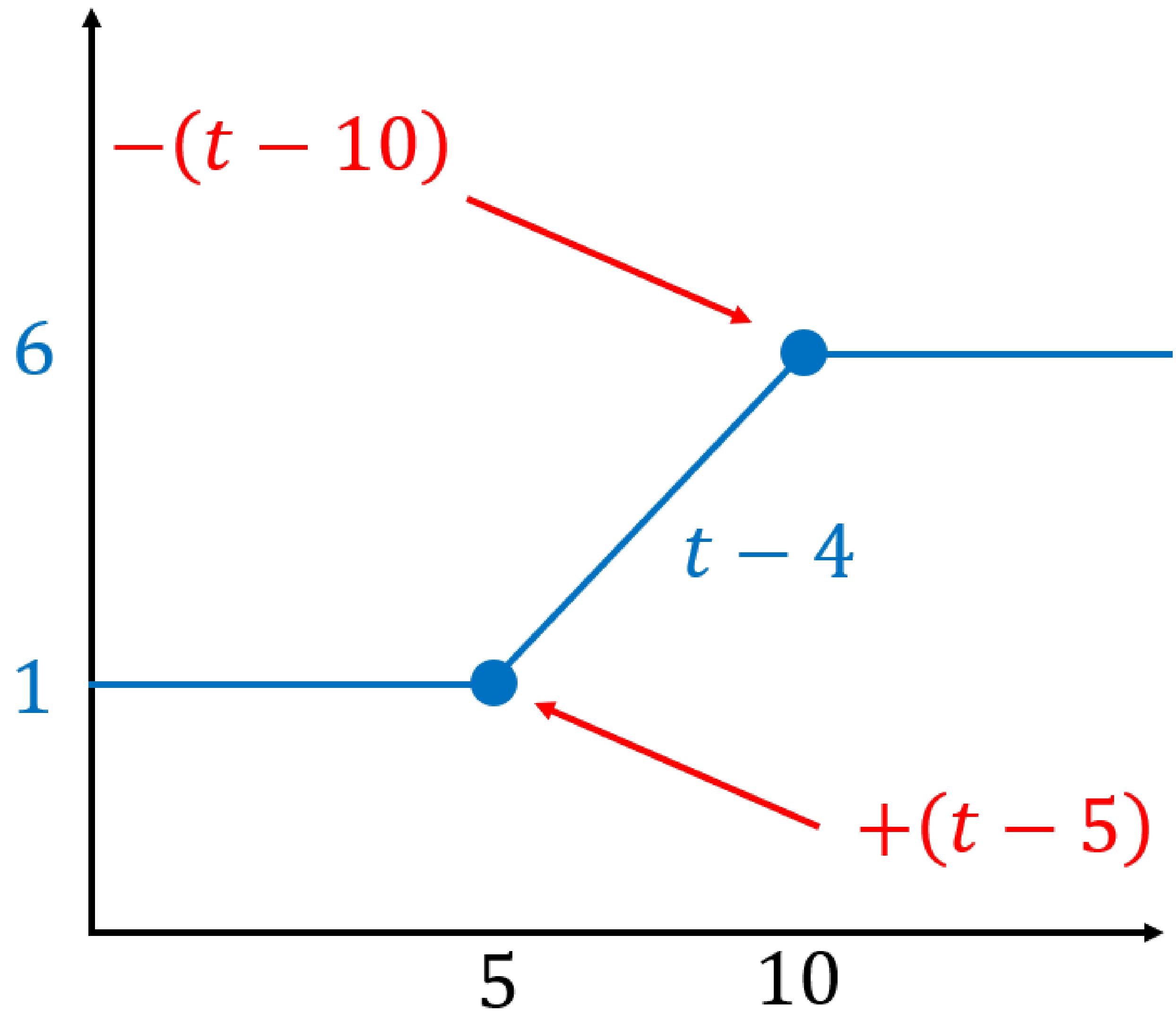
Example 3:

$$\mathcal{L}\left\{\cos(2t-6)u_3(t)\right\}$$

$$\mathcal{L}\left\{\cos(2t-6)u_3(t)\right\} = \mathcal{L}\left\{\cos(2(t-3))u_3(t)\right\} = e^{-3s}\mathcal{L}\left\{\cos(2t)\right\} = e^{-3s}\left(\frac{s}{s^2+4}\right)$$

Example 4:

$$\text{Find } \mathcal{L}\left\{g(t)\right\} \text{ where } g(t) = \begin{cases} 1 & \text{if } t < 5 \\ t-4 & \text{if } 5 \le t < 10 \\ 6 & \text{if } t \ge 10 \end{cases}$$



STEP 1: Write g in terms of u_c

Start at 1

At $t = 5$ jump by $t - 4 - 1 = t - 5$

At $t = 10$ jump by $6 - (t-4) = 10 - t = -(t-10)$

(Important to write in terms of $t - 10$)

$$g(t) = 1 + (t-5)u_5(t) - (t-10)u_{10}(t)$$

STEP 2: Take Laplace transforms

$$\begin{aligned}
\mathcal{L}\left\{g(t)\right\} =& \mathcal{L}\left\{1\right\} + \mathcal{L}\left\{(t-5)u_5(t)\right\} - \mathcal{L}\left\{(t-10)u_{10}(t)\right\} \\
=& \left(\frac{1}{s}\right) + e^{-5s}\mathcal{L}\left\{t\right\} - e^{-10s}\mathcal{L}\left\{t\right\} \\
=& \left(\frac{1}{s}\right) + e^{-5s}\left(\frac{1!}{s^2}\right) - e^{-10s}\left(\frac{1!}{s^2}\right) \\
=& \left(\frac{1}{s}\right) + \left(\frac{e^{-5s}}{s^2}\right) - \left(\frac{e^{-10s}}{s^2}\right) \\
=& \left(\frac{1}{s}\right) + \left(\frac{1}{s^2}\right)\left(e^{-5s} - e^{-10s}\right)
\end{aligned}$$

3. Reverse Way

It's important to be also able to do this in reverse

Note: If you read the above formula from right to left, you get

Fact:

$$e^{-cs}\mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{f(t-c)u_c(t)\right\}$$

In other words, an exponential term *outside* the laplace term causes the function to shift and jump

Example 5:

Find a function whose Laplace transform is

$$e^{-5s}\left(\frac{6}{s^4}\right)$$

$$e^{-5s}\left(\frac{6}{s^4}\right) = e^{-5s}\left(\frac{3!}{s^4}\right) = e^{-5s}\mathcal{L}\left\{t^3\right\} = \mathcal{L}\left\{(t-5)^3 u_5(t)\right\}$$

$$f(t) = (t-5)^3\, u_5(t)$$

Example 6:

Find a function whose Laplace transform is

$$\left(\frac{8}{s^2+16}\right)e^{-7s}$$

$$e^{-7s}\left(\frac{8}{s^2+16}\right) = e^{-7s}\,2\left(\frac{4}{s^2+4^2}\right) = e^{-7s}\mathcal{L}\left\{2\sin(4t)\right\} = \mathcal{L}\left\{2\sin\left(4(t-7)\right)u_7(t)\right\}$$

$$f(t) = 2\sin\left(4(t-7)\right)u_7(t)$$

Next time: What happens if the exponential term is *inside* the Laplace transform?