LECTURE: STEP FUNCTIONS (II)

1. Shifting

If the exponential term is *inside* the Laplace Transform, we get shifts.

Example 1:

$$\mathcal{L}\left\{e^{3t}\cos(2t)\right\}$$

$$\text{Let } F(s) = \mathcal{L}\left\{\cos(2t)\right\} = \frac{s}{s^2+4}$$

$$\begin{aligned}
\mathcal{L}\left\{e^{3t}\cos(2t)\right\} &= \int_0^\infty e^{3t}\cos(2t)e^{-st}dt \\
&= \int_0^\infty \cos(2t)e^{-(s-3)t}dt \\
&= \mathcal{L}\left\{\cos(2t)\right\} \text{ but evaluated at } s-3 \\
&= F(s-3) \\
&= \frac{s-3}{(s-3)^2+4}
\end{aligned}$$

Fact:

$$\mathcal{L}\left\{e^{3t}\cos(2t)\right\} = \frac{s-3}{(s-3)^2+4}$$

That is, exponential terms just shift the Laplace transform.

Example 2:

$$\mathcal{L}\left\{e^{-t}t^2\right\}$$

$$F(s) = \mathcal{L}\left\{t^2\right\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$\text{Hence } \mathcal{L}\left\{e^{-t}t^2\right\} = F(s-(-1)) = F(s+1) = \frac{2}{(s+1)^3}$$

2. Reverse Way

Once again, it's important to do this in reverse:

Example 3:

Find a function whose Laplace transform is

$$\frac{1}{(s-2)^2+1}$$

This is a shifted version by 2 units of

$$\frac{1}{s^2+1} = \mathcal{L}\left\{\sin(t)\right\}$$

Since exponential terms shift Laplace transforms, we get

$$\frac{1}{(s-2)^2+1} = \mathcal{L}\left\{e^{2t}\sin(t)\right\}$$

Answer: $f(t) = e^{2t}\sin(t)$

Example 4:

Find a function whose Laplace transform is

$$\frac{6}{(s+3)^4}$$

This is a shifted version of

$$\frac{6}{s^4} = \frac{3!}{s^4} = \mathcal{L}\left\{t^3\right\}$$

Since we shifted by $s+3 = s-(-3)$ we have

$$\mathcal{L}\left\{e^{-3t}t^3\right\} = \frac{6}{(s-(-3))^4} = \frac{6}{(s+3)^4}$$

$$\text{Answer: } f(t) = e^{-3t}\left(t^3\right)$$

3. Summary: Four Cases

Case 1: Laplace transforms of jumps: an exponential term pops out

$$\mathcal{L}\left\{(t-2)^3 u_2(t)\right\} = e^{-2s}\mathcal{L}\left\{t^3\right\} = e^{-2s}\left(\frac{6}{s^4}\right)$$

Case 2: The exponential is outside: Shift and add a jump

$$e^{-5s}\mathcal{L}\left\{t^3\right\} = \mathcal{L}\left\{(t-5)^3\, u_5(t)\right\}$$

Case 3: The exponential is inside: Shift the Laplace transform

$$\mathcal{L}\left\{e^{3t}\cos(2t)\right\} = \frac{s-3}{(s-3)^2+4}$$

Case 4: Laplace transform is shifted: Add an exponential term inside

$$\frac{1}{(s-2)^2+1} = \mathcal{L}\left\{e^{2t}\sin(t)\right\}$$

4. Completing the Square

Example 5:

Find a function whose Laplace transform is

$$\frac{1}{s^2+4s+5}$$

Need to complete the square

$$\frac{1}{s^2+4s+5} = \frac{1}{(s+2)^2-4+5} = \frac{1}{(s+2)^2+1}$$

This is a shifted version by -2 units of $\frac{1}{s^2+1} = \mathcal{L}\{\sin(t)\}$, therefore

Answer: $f(t) = e^{-2t}\sin(t)$

Example 6:

Find a function whose Laplace transform is

$$\frac{s}{s^2-6s+13}$$

Complete the square

$$\begin{aligned}\frac{s}{s^2-6s+13} &= \frac{s}{(s-3)^2-9+13} \\ &= \frac{s}{(s-3)^2+4} \qquad \text{Write in terms of } s-3 \\ &= \frac{(s-3)+3}{(s-3)^2+4} \\ &= \frac{s-3}{(s-3)^2+4} + \frac{3}{(s-3)^2+4}\end{aligned}$$

This is a shifted version by 3 units of

$$\frac{s}{s^2+4} + \frac{3}{s^2+4} = \frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right) = \mathcal{L}\left\{\cos(2t) + \frac{3}{2}\sin(2t)\right\}$$

$$\textbf{Answer: } f(t) = e^{3t}\left(\cos(2t) + \frac{3}{2}\sin(2t)\right)$$

5. Putting everything together

Finally, we can combine this with the jump property:

Example 7:

Find a function whose Laplace transform is

$$\frac{3(s-2)e^{-3s}}{s^2-4s+5}$$

STEP 1: First focus on

$$\frac{3(s-2)}{s^2-4s+5}$$

Complete the square on the bottom: $s^2-4s+5=(s-2)^2+1$

$$\frac{3(s-2)e^{-3s}}{(s-2)^2+1} = e^{-3s}\left(\frac{3(s-2)}{(s-2)^2+1}\right)$$

The terms in parentheses is a shifted version by 2 units of

$$\frac{3s}{s^2+1} = \mathcal{L}\left\{3\cos(t)\right\}$$

$$\text{Therefore } \frac{3(s-2)}{(s-2)^2+1} = \mathcal{L}\left\{e^{2t}\,3\cos(t)\right\}$$

STEP 2: Now focus on the exponential term

$$e^{-3s}\left(\frac{3(s-2)}{(s-2)^2+1}\right) = e^{-3s}\mathcal{L}\left\{3e^{2t}\cos(t)\right\} = \mathcal{L}\left\{3e^{2(t-3)}\cos(t-3)u_3(t)\right\}$$

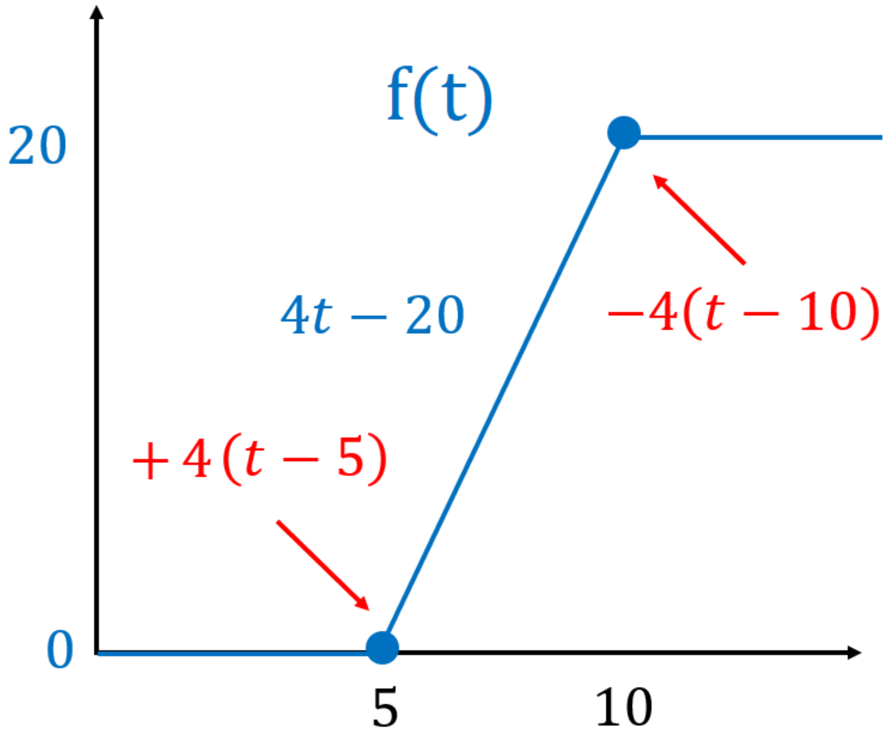
$$\textbf{Answer:}\quad f(t) = 3e^{2(t-3)}\cos(t-3)u_3(t)$$

6. ODE with Jumps

As an application, let's solve ODE with jumps

Example 8:

$$\begin{cases} y'' + 4y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} 0 & \text{if } 0 \le t < 5 \\ 4t - 20 & \text{if } 5 \le t < 10 \\ 20 & \text{if } t \ge 10 \end{cases}$$



STEP 1: Write $f(t)$ in terms of u_c

Start at 0

At $t = 5$, jump by $4t - 20 = 4(t-5)$

At $t = 10$, jump by $20 - (4t - 20) = 40 - 4t = 4(10 - t) = -4(t - 10)$

$$f(t) = 4(t-5)u_5 - 4(t-10)u_{10}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} =& \mathcal{L}\{4(t-5)u_5(t)\} - \mathcal{L}\{4(t-10)u_{10}(t)\} \\ =& e^{-5s}\mathcal{L}\{4t\} - e^{-10s}\mathcal{L}\{4t\} \\ =& \frac{4}{s^2}\left(e^{-5s} - e^{-10s}\right)\end{aligned}$$

STEP 2: Next time