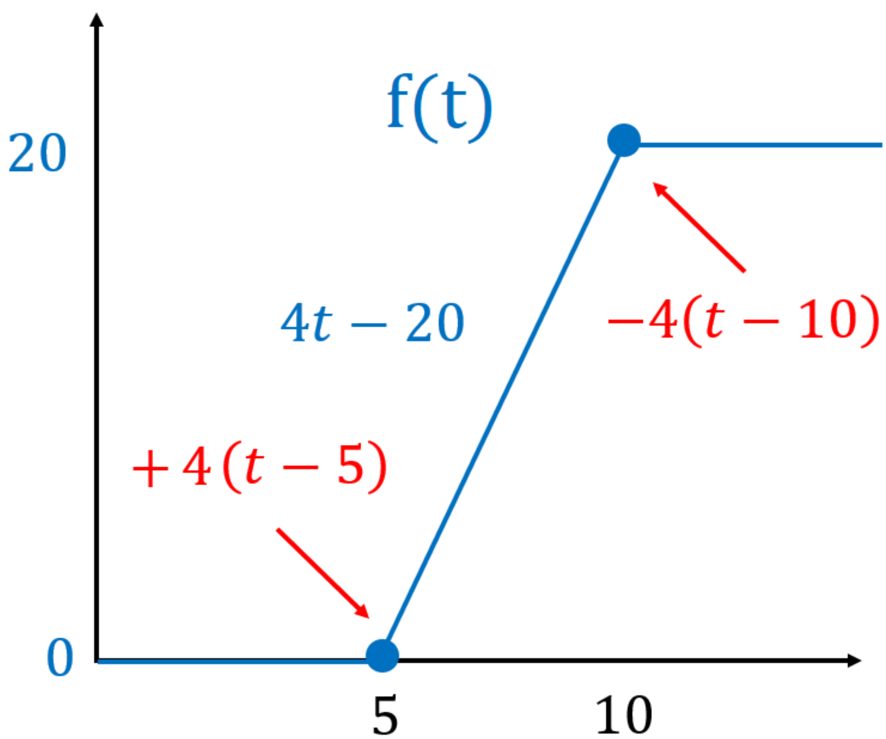
LECTURE: ODE WITH JUMPS

1. ODE with Jumps

Example 1:

$$\begin{cases} y'' + 4y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} 0 & \text{if } 0 \le t < 5 \\ 4t - 20 & \text{if } 5 \le t < 10 \\ 20 & \text{if } t \ge 10 \end{cases}$$



STEP 1: Last time: By writing $f(t)$ in terms of u_c and taking Laplace transforms, we got

$$\mathcal{L}\left\{f(t)\right\} = \frac{4}{s^2}\left(e^{-5s} - e^{-10s}\right)$$

STEP 2: Take Laplace Transforms of the ODE

$$\begin{aligned}
\mathcal{L}\left\{y''\right\} + 4\mathcal{L}\left\{y\right\} =& \mathcal{L}\left\{f(t)\right\} \\
\left(s^2\mathcal{L}\left\{y\right\} - \underbrace{sy(0) - y'(0)}_{0}\right) + 4\mathcal{L}\left\{y\right\} =& \frac{4}{s^2}\left(e^{-5s} - e^{-10s}\right) \\
\left(s^2+4\right)\mathcal{L}\left\{y\right\} =& \frac{4}{s^2}\left(e^{-5s} - e^{-10s}\right) \\
\mathcal{L}\left\{y\right\} =& \frac{4}{s^2\left(s^2+4\right)}\left(e^{-5s} - e^{-10s}\right)
\end{aligned}$$

STEP 3: Partial Fractions

Since $\frac{1}{s^2}$ is *repeated* have to guess:

$$\begin{aligned}
\frac{4}{s^2\left(s^2+4\right)} =& \frac{As+B}{s^2} + \frac{Cs+D}{s^2+4} \\
=& \frac{As(s^2+4) + B(s^2+4) + (Cs+D)s^2}{s^2(s^2+4)} \\
=& \frac{As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2}{s^2(s^2+4)} \\
\frac{0s^3 + 0s^2 + 0s + 4}{s^2\left(s^2+4\right)} =& \frac{(A+C)s^3 + (B+D)s^2 + 4As + 4B}{s^2(s^2+4)}
\end{aligned}$$

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 4A = 0 \\ 4B = 4 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 1 \\ C = -A = 0 \\ D = -B = -1 \end{cases}$$

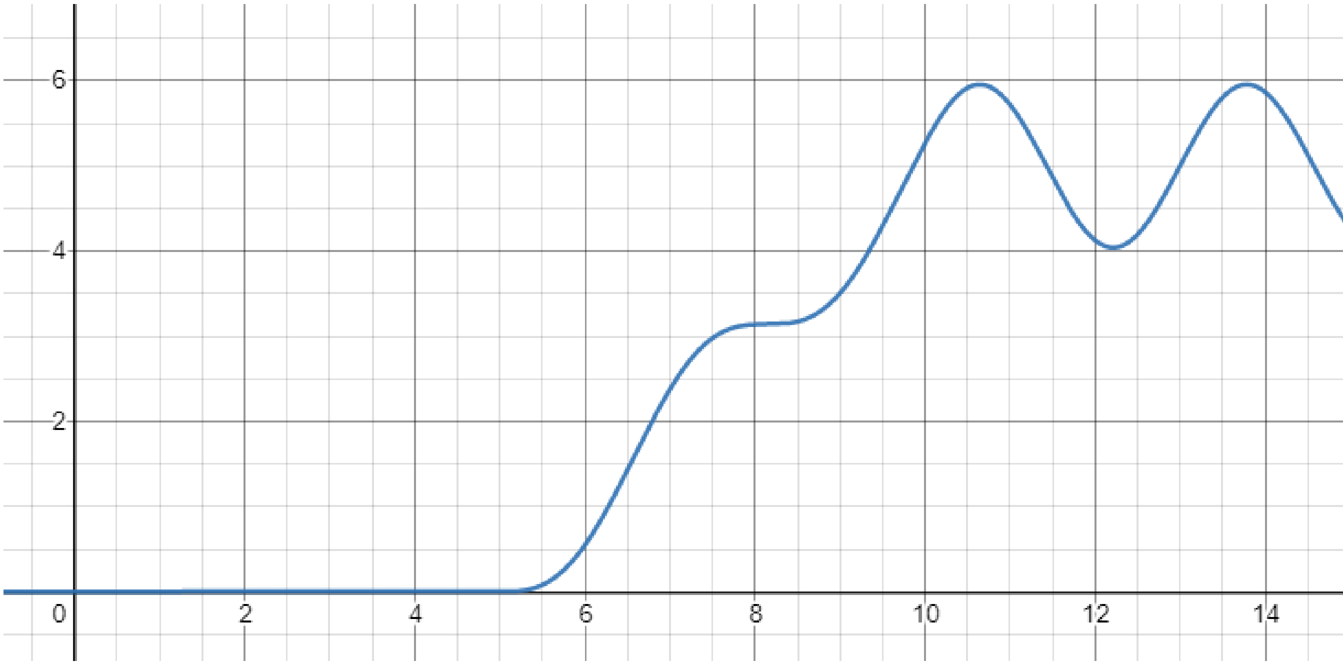
$$\frac{4}{s^2\left(s^2+4\right)} = \frac{0}{s} + \frac{1}{s^2} + \frac{0s-1}{s^2+4} = \frac{1}{s^2} - \frac{1}{s^2+4}$$

STEP 3:

$$\begin{aligned} \mathcal{L}\left\{y\right\} &= \left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)\left(e^{-5s} - e^{-10s}\right) \\ &= \mathcal{L}\left\{t - \frac{1}{2}\sin(2t)\right\}\left(e^{-5s} - e^{-10s}\right) \\ &= \mathcal{L}\left\{h(t)\right\}\left(e^{-5s} - e^{-10s}\right) \qquad \text{where } h(t) = t - \frac{1}{2}\sin(2t) \\ &= \mathcal{L}\left\{h(t-5)u_5(t) - h\left(t-10\right)u_{10}(t)\right\} \end{aligned}$$

STEP 4: Answer:

$$\begin{aligned} y &= h(t-5)u_5(t) - h(t-10)u_{10}(t) \\ \text{where } h(t) &= t - \frac{1}{2}\sin(2t) \end{aligned}$$



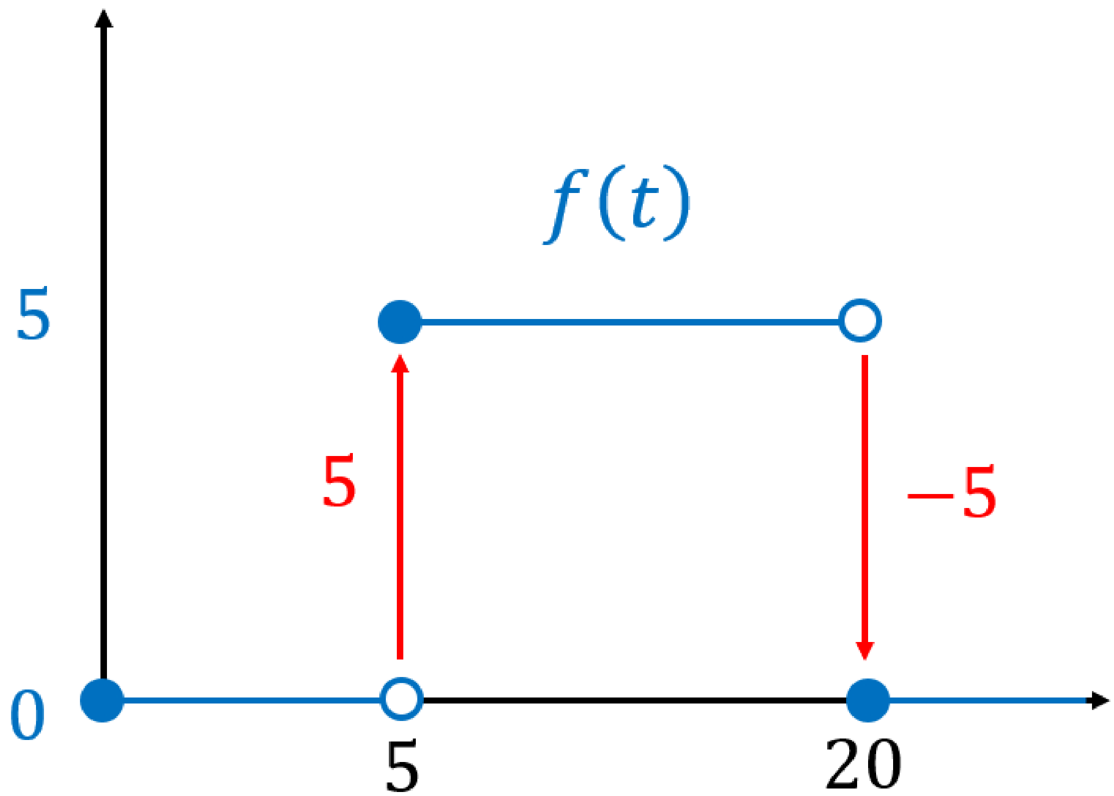
You can spot the three different regimes:

- On $[0,5]$ y is constant
- On $[5,10]$ y is in a transition phase
- On $[10,\infty)$ y is sinusoidal

2. ODE with Shifts

Example 2:

$$\begin{cases} y'' + 4y' + 5y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 5 \\ 5 & \text{if } 5 \leq t < 20 \\ 0 & \text{if } t \geq 20 \end{cases}$$



STEP 1: Start at 0, jump by 5 at $t = 5$, and jump by -5 at $t = 20$

$$f(t) = 5u_5(t) - 5u_{20}(t)$$

$$\mathcal{L}\left\{f(t)\right\} = 5\left(\frac{e^{-5s}}{s}\right) - 5\left(\frac{e^{-20s}}{s}\right) = \left(\frac{5}{s}\right)\left(e^{-5s} - e^{-20s}\right)$$

STEP 2:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\left(s^2\mathcal{L}\{y\} - \underbrace{sy(0) - y'(0)}_{0}\right) + 4\left(s\mathcal{L}\{y\} - \underbrace{y(0)}_{0}\right) + 5\mathcal{L}\{y\} = \left(\frac{5}{s}\right)\left(e^{-5s} - e^{-20s}\right)$$

$$\left(s^2 + 4s + 5\right)\mathcal{L}\{y\} = \left(\frac{5}{s}\right)\left(e^{-5s} - e^{-20s}\right)$$

$$\mathcal{L}\{y\} = \frac{5}{s(s^2 + 4s + 5)}\left(e^{-5s} - e^{-20s}\right)$$

STEP 3: Partial Fractions

$$\begin{aligned}
\frac{5}{s\left(s^2 + 4s + 5\right)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5} \\
&= \frac{A(s^2 + 4s + 5) + (Bs + C)s}{s(s^2 + 4s + 5)} \\
&= \frac{As^2 + 4As + 5A + Bs^2 + Cs}{s(s^2 + 4s + 5)} \\
\frac{0s^2 + 0s + 5}{s\left(s^2 + 4s + 5\right)} &= \frac{(A + B)s^2 + (4A + C)s + 5A}{s(s^2 + 4s + 5)}
\end{aligned}$$

$$\begin{cases} A + B = 0 \\ 4A + C = 0 \\ 5A = 5 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -A = -1 \\ C = -4A = -4 \end{cases}$$

$$\frac{5}{s\left(s^2 + 4s + 5\right)} = \left(\frac{1}{s}\right) + \left(\frac{-s - 4}{s^2 + 4s + 5}\right)$$

STEP 4:

$$\mathcal{L}\{y\} = \left(\frac{1}{s} - \frac{s + 4}{s^2 + 4s + 5}\right)\left(e^{-5s} - e^{-20s}\right)$$

Since $\frac{1}{s} = \mathcal{L}\left\{1\right\}$ we just need to figure out $\frac{s+4}{s^2+4s+5} = \mathcal{L}\left\{?\right\}$

$$\frac{s+4}{s^2+4s+5} = \frac{s+4}{(s+2)^2+1} = \frac{s+2+2}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1}$$

This is a shifted version by -2 units of

$$\frac{s}{s^2+1} + \frac{2}{s^2+1} = \mathcal{L}\left\{\cos(t) + 2\sin(t)\right\}$$

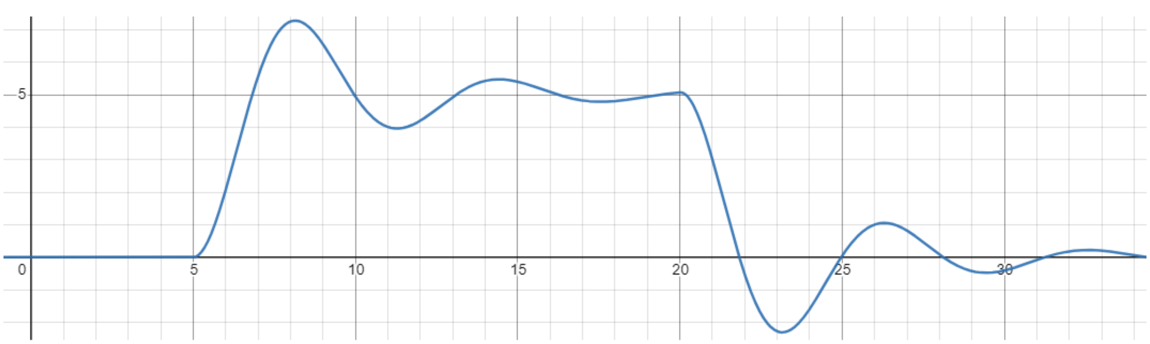
$$\frac{s+4}{s^2+4s+5} = \mathcal{L}\left\{e^{-2t}\left(\cos(t) + 2\sin(t)\right)\right\}$$

STEP 5: Back to the ODE

$$\begin{aligned}
\mathcal{L}\left\{y\right\} &= \left(\frac{1}{s} - \frac{s+4}{s^2+4s+5}\right)\left(e^{-5s} - e^{-20s}\right) \\
&= \mathcal{L}\left\{1 - e^{-2t}\left(\cos(t) + 2\sin(t)\right)\right\}\left(e^{-5s} - e^{-20s}\right) \\
&= \mathcal{L}\left\{h(t)\right\}\left(e^{-5s} - e^{-20s}\right) \qquad \text{where } h(t) = 1 - e^{-2t}\left(\cos(t) - 2\sin(t)\right) \\
&= \mathcal{L}\left\{h(t-5)u_5(t) - h(t-20)u_{20}(t)\right\}
\end{aligned}$$

STEP 6: Solution

$$\begin{aligned}
y &= h(t-5)u_5(t) - h(t-20)u_{20}(t) \\
\text{where } h(t) &= 1 - e^{-2t}\left(\cos(t) - 2\sin(t)\right)
\end{aligned}$$



Notice again the three different regimes:

On $[0, 5]$ y is 0

On $[5, 20]$ y is in a transition phase

On $[20, \infty)$ y is sinusoidal with damping