LECTURE: DIRAC DELTA

Goal: How to model a unit impulse, like being pinched by someone, or being struck by lightning. You can do this with Dirac Delta

1. Dirac and roll

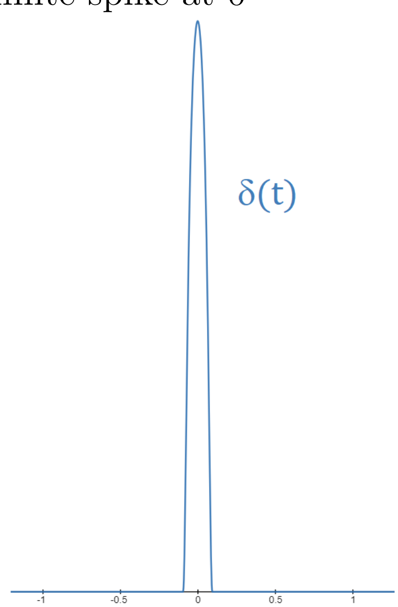
Fact:

There is a "function" $\delta(t)$ such that:

(1) $\delta(t) = 0$ except at $t = 0$, where $\delta(0) = \infty$

(2) $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Think of $\delta(t)$ as an infinite spike at 0



Note: The Dirac Delta **NOT** a function! Because if f is a *function* that's 0 almost everywhere, then $\int_{-\infty}^{\infty} f(t)dt = 0$ not 1

That's why the Dirac Delta is sometimes called a **distribution**

Note: This has an interesting probability interpretation: The Dirac Delta models a game where most of the time you lose (get 0 dollars) but there is a very small chance (probability 0) to win ∞ dollars, in such a way that the expected payout is $\int_{-\infty}^{\infty} \delta(t)dt = 1$

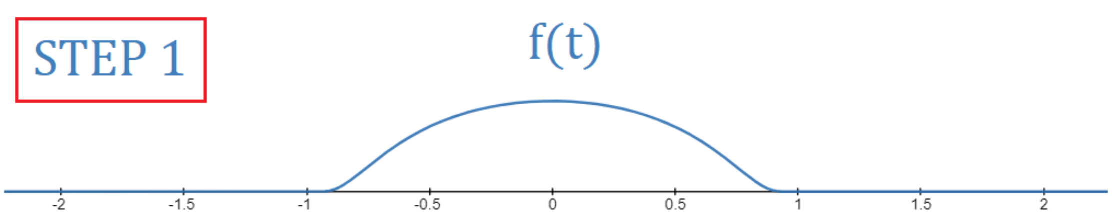
Aside: If you want to know what the derivative of the Dirac Delta is, check out this video:

Video: Dirac Delta Derivative

2. Dirac Delta Construction

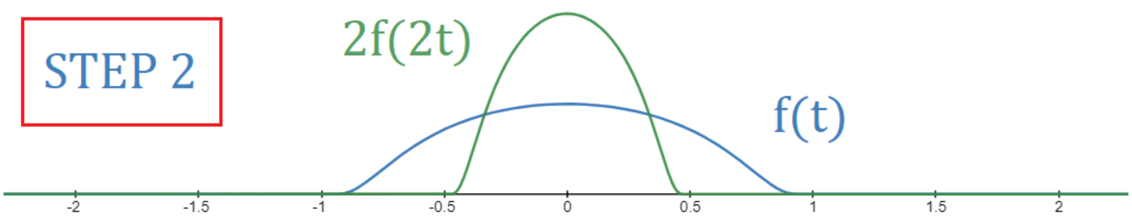
How to construct the Dirac Delta? This is done in stages.

STEP 1: Start with any smooth function $f(t)$ that is 0 outside $[-1, 1]$ and positive on $(-1, 1)$ and such that $\int_{-\infty}^{\infty} f(t)dt = 1$



There are many choices for f, but one that is often used is the bump function $f(t) = e^{\frac{1}{1-(t^2)}}$ on $(-1, 1)$ and 0 outside $[-1, 1]$

STEP 2: Given $f(t)$ as in **STEP 1** consider $2f(2t)$ This stretches out f vertically and compresses it horizontally, to make it "spikier"



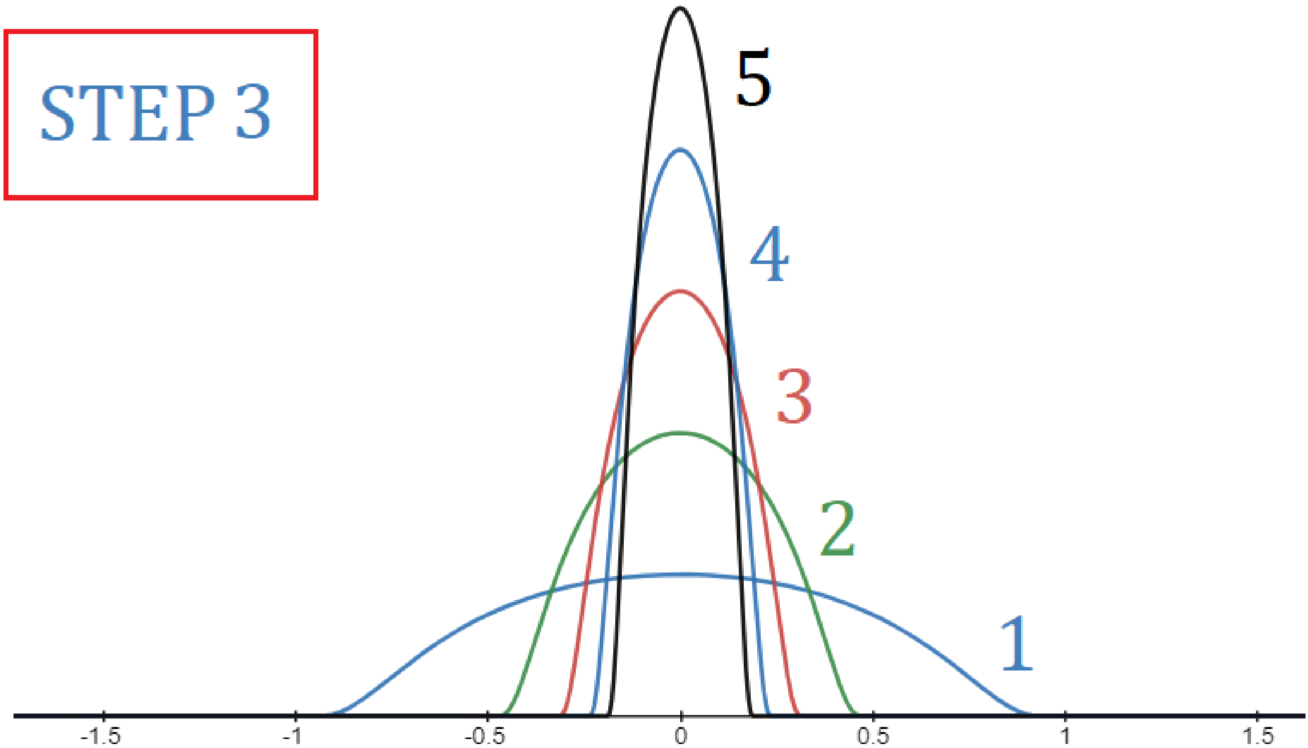
Then $2f(2t)$ is zero outside $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and also

$$\int_{-\infty}^{\infty} 2f(2t)dt = \int_{-\infty}^{\infty} f(u)du = 1$$

Here we used $u = 2t$ so $du = 2dt$ and we also used $\int_{-\infty}^{\infty} f = 1$

So the new function still has integral 1, but is nonzero on a smaller set.

STEP 3: There's nothing special about 2, and in fact we can now consider $3f(3t)$, $4f(4t)$, $5f(5t)$, ...



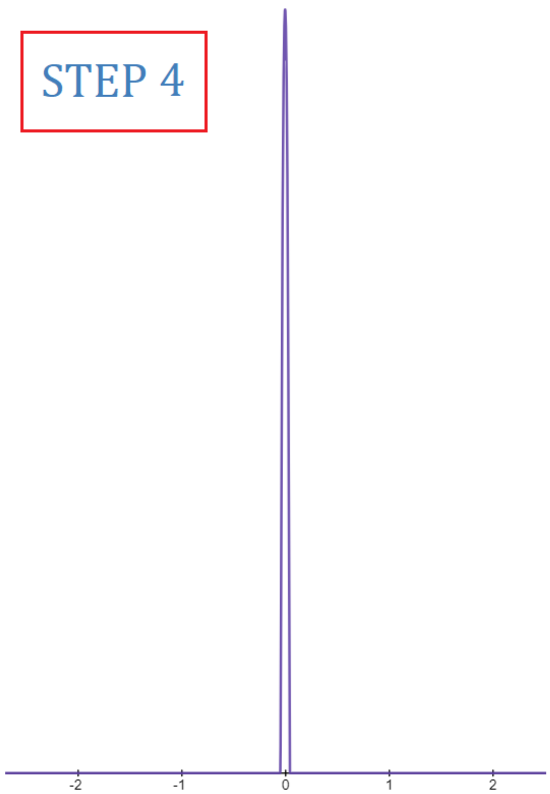
The functions become successively narrower and spikier, but their integrals are 1.

STEP 4: The Dirac Delta is just the limit of those functions:

Definition:

$$\delta(t) = \lim_{n \to \infty} n f(nt)$$

In other words, just think of $\delta(t) = 1000 f(1000t)$ where that 1000 is replaced by a very large number.



Then, we can see that $\delta(t) = 0$ except at $t = 0$, where $\delta(0) = \infty$. And since, at each step, the integrals are 1, it follows that $\int_{-\infty}^{\infty} \delta(t)dt = 1$

3. Dirac Delta Laplace Transform

Question: What is $\mathcal{L}\{\delta(t)\}$?

This is related to the most important property of Dirac Delta:

Fact:

For *any* function $f(t)$, we have

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

Intuitively this makes sense because $\delta(t) = 0$ except for $t = 0$, so $\delta(t)f(t)$ "concentrates" f to become $f(0)$. In fact, in advanced math classes, this is sometimes taken as the definition of the Dirac Delta.

Note: An optional proof is given at the end of the notes

Using this property, we can easily find the Laplace transform of $\delta(t)$:

Fact:

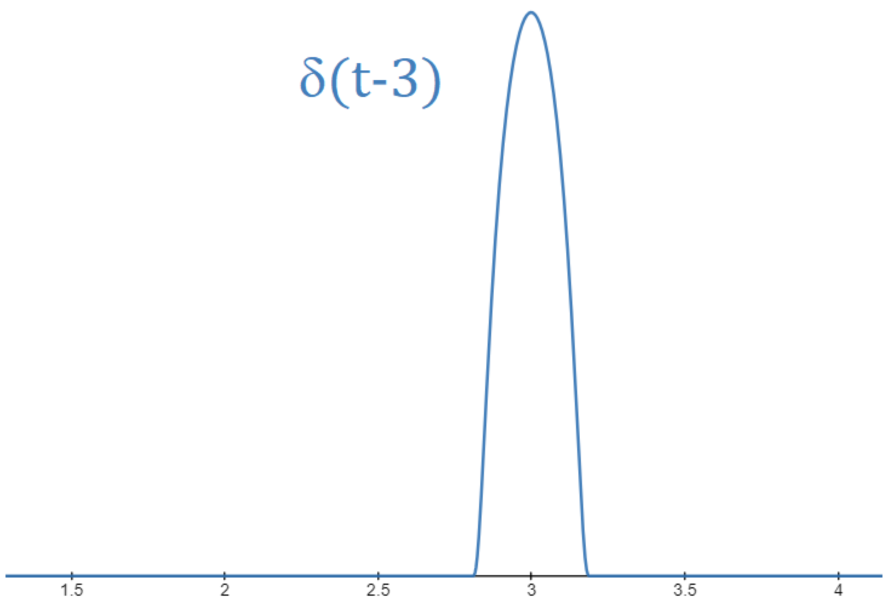
$$\mathcal{L}\left\{\delta(t)\right\} = 1$$

Why? Using the definition of the Laplace transform, $\delta(t) = 0$ for $t < 0$ and the Fact above, we get

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{0}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) \underbrace{e^{-st}}_{f(t)} dt = f(0) = e^{-s(0)} = 1$$

This is interesting because we know $\mathcal{L}\left\{1\right\} = \frac{1}{s}$ but we never found a function so far whose Laplace transform is 1

Other points: Similarly, $\delta(t-3)$ is a Dirac Delta at $t = 3$



The property becomes $\int_{-\infty}^{\infty} \delta(t-3) f(t) dt = f(3)$ for all f and therefore

$$\mathcal{L}\left\{\delta(t-3)\right\} = \int_0^{\infty} \delta(t-3) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t-3) \underbrace{e^{-st}}_{f(t)} dt = f(3) = e^{-s(3)} = e^{-3s}$$

Fact:

$$\mathcal{L}\left\{\delta(t-c)\right\} = e^{-cs}$$

Example 1:

$$\mathcal{L}\left\{4\delta(t) - 2\delta(t-3) + 5\delta(t-6)\right\} = 4 - 2e^{-3s} + 5e^{-6s}$$

4. ODE with Dirac Delta

And using this, we can solve ODE with Dirac Delta:

Example 2:

$$\begin{cases} y'' - 2y' + y = \delta(t-5) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

STEP 1: Take Laplace Transforms

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-5)\}$$

$$\left(s^2\mathcal{L}\{y\} - \underbrace{sy(0) - y'(0)}_{0}\right) - 2\left(s\mathcal{L}\{y\} - \underbrace{y(0)}_{0}\right) + \mathcal{L}\{y\} = e^{-5s}$$

$$\left(s^2 - 2s + 1\right)\mathcal{L}\{y\} = e^{-5s}$$

$$\mathcal{L}\{y\} = \left(\frac{1}{s^2 - 2s + 1}\right)e^{-5s}$$

STEP 2: Look at

$$\frac{1}{s^2 - 2s + 1} = \frac{1}{(s-1)^2}$$

This is a shifted version by 1 unit of

$$\frac{1}{s^2} = \mathcal{L}\{t\}$$

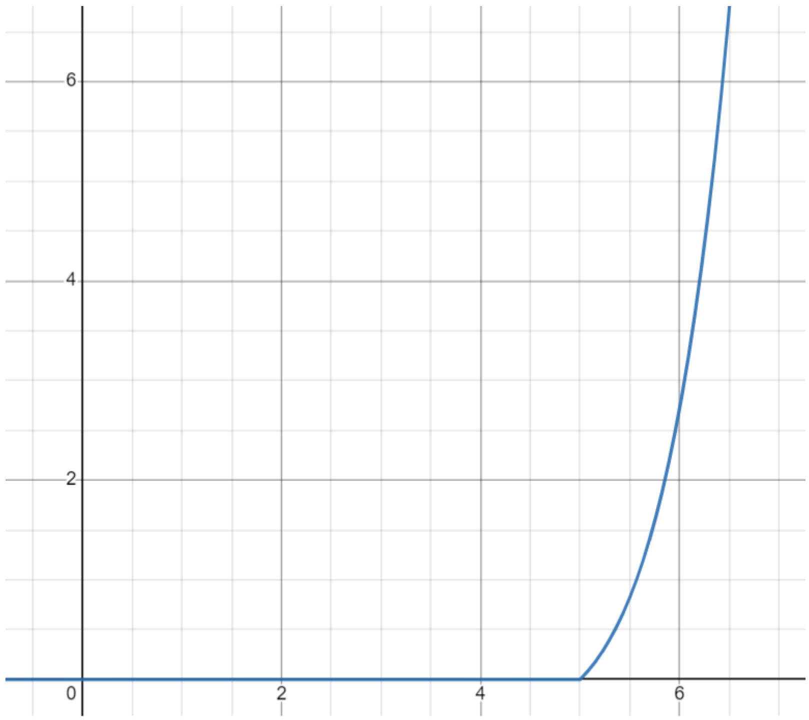
$$\text{Therefore: } \frac{1}{(s-1)^2} = \mathcal{L}\left\{e^t\, t\right\} = \mathcal{L}\left\{te^t\right\}$$

STEP 3:

$$\mathcal{L}\{y\} = \left(\frac{1}{s^2 - 2s + 1}\right)e^{-5s} = \mathcal{L}\left\{te^t\right\}e^{-5s} = \mathcal{L}\left\{(t-5)\,e^{t-5}u_5(t)\right\}$$

STEP 4: Answer:

$$y = (t-5)e^{t-5}u_5(t)$$



Notice how the solution starts out at 0 but is coming to life at $t = 5$

Example 3:

$$\begin{cases} y'' + 3y' + 2y = \delta(t-3) \\ y(0) = 2 \\ y'(0) = -3 \end{cases}$$

STEP 1: Laplace transforms

$$\begin{aligned}\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} =& \mathcal{L}\{\delta(t-3)\} \\ \left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} =& e^{-3s} \\ \left(s^2\mathcal{L}\{y\} - 2s + 3\right) + (3s\mathcal{L}\{y\} - 6) + 2\mathcal{L}\{y\} =& e^{-3s} \\ \left(s^2 + 3s + 2\right)\mathcal{L}\{y\} =& 2s + 3 + e^{-3s} \\ \mathcal{L}\{y\} =& \left(\frac{2s+3}{s^2+3s+2}\right) + \left(\left(\frac{1}{s^2+3s+2}\right)e^{-3s}\right)\end{aligned}$$

STEP 2: Partial Fractions

Note: Here you have to do two partial fractions

First term: Since $s^2 + 3s + 2 = (s+1)(s+2)$ we get

$$\frac{2s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)} = \frac{(A+B)s + (2A+B)}{(s+1)(s+2)}$$

$$\begin{cases} A + B = 2 \\ 2A + B = 3 \end{cases}$$

Using $B = 2 - A$ and plugging this into the second equation you get $A = 1$ and $B = 1$

$$\frac{2s+3}{s^2+3s+2} = \frac{1}{s+1} + \frac{1}{s+2}$$

Second Term:

$$\frac{1}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{(A+B)s + (2A+B)}{(s+1)(s+2)}$$

$$\begin{cases} A + B = 0 \\ 2A + B = 1 \end{cases}$$

Using $B = -A$ and plugging this into the second equation you get $A = 1$ and $B = -1$

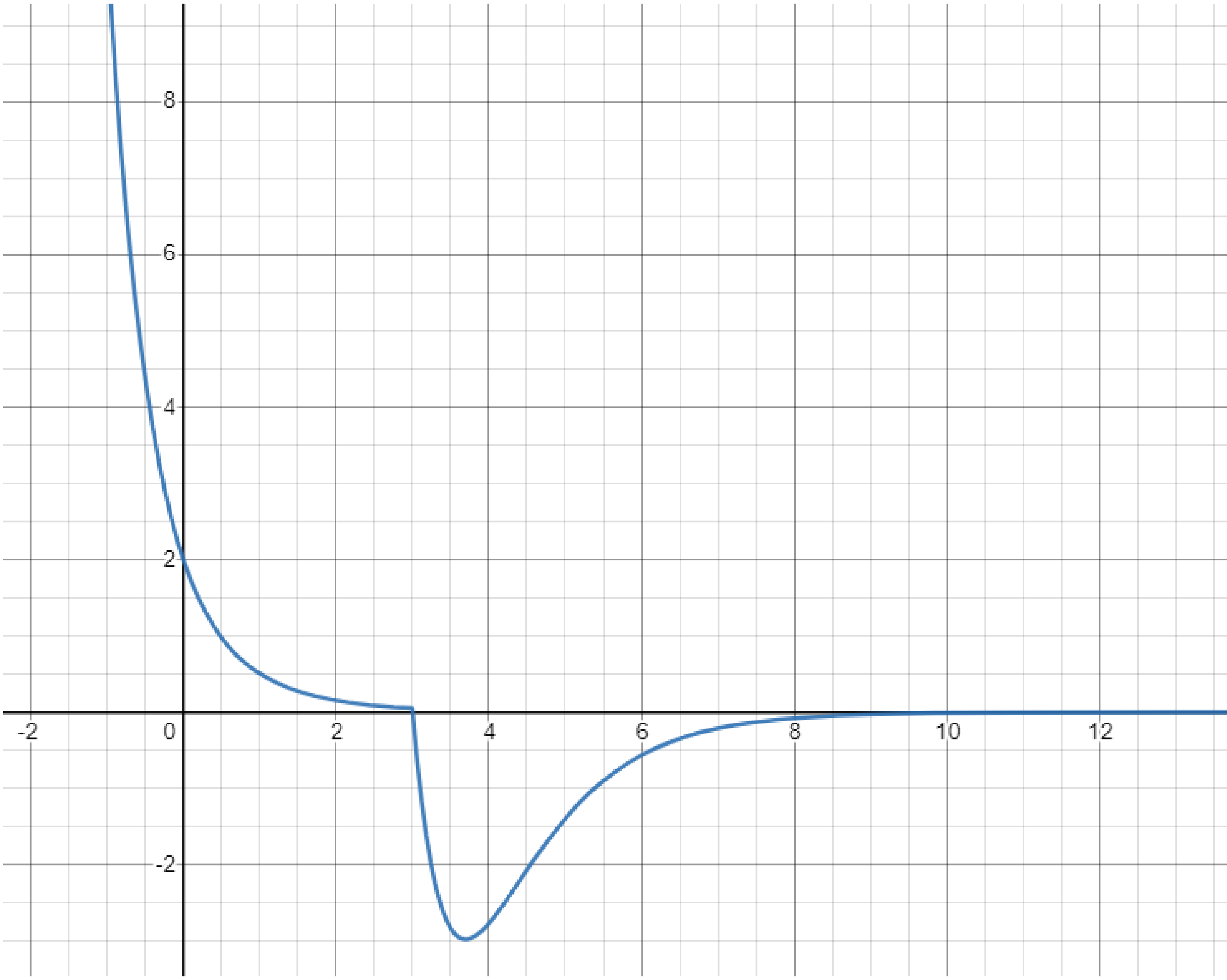
$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

STEP 3:

$$\begin{aligned}
\mathcal{L}\{y\} &= \left(\frac{2s+3}{s^2+3s+2}\right) + \left[\left(\frac{1}{s^2+3s+2}\right)e^{-3s}\right] \\
&= \left(\frac{1}{s+1} + \frac{1}{s+2}\right) + \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-3s}\right] \\
&= \mathcal{L}\left\{e^{-t} + e^{-2t}\right\} + \mathcal{L}\left\{e^{-t} - e^{-2t}\right\} e^{-3s} \\
&= \mathcal{L}\left\{e^{-t} + e^{-2t} + \left(e^{-(t-3)} - e^{-2(t-3)}\right) u_3(t)\right\}
\end{aligned}$$

STEP 4: Answer:

$$y = e^{-t} + e^{-2t} + \left(e^{-(t-3)} - e^{-2(t-3)}\right) u_3(t)$$



Again the function is getting a kick at $t = 3$

5. Appendix: Proof of Fact

Fact:

For *any* function $g(t)$, we have

$$\int_{-\infty}^{\infty} \delta(t)g(t)dt = g(0)$$

(Here we use g because we'll need f for something else)

Why? **STEP 1:** First of all, recall that

$$\delta(t) = \lim_{n\to\infty} nf(nt)$$

Where f is the bump function from **STEP 1** of the Dirac Delta Construction. In particular remember $\int_{-\infty}^{\infty} f(t)dt = 1$

STEP 2: For every n we have

$$\int_{-\infty}^{\infty} nf(nt)g(t)dt = \int_{-\infty}^{\infty} f(u)g\left(\frac{u}{n}\right)du$$

Here we used $u = nt$ so $du = ndt$ and $t = \frac{u}{n}$

STEP 3: Now take $\lim_{n\to\infty}$ on both sides:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} nf(nt)g(t)dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(u)g\left(\frac{u}{n}\right)du$$

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} nf(nt)g(t)dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} f(u)g\left(\frac{u}{n}\right)du$$

$$\int_{-\infty}^{\infty} \left(\lim_{n\to\infty} nf(nt)\right)g(t)dt = \int_{-\infty}^{\infty} f(u)\left(\lim_{n\to\infty} g\left(\frac{u}{n}\right)\right)du$$

STEP 4: On the other hand, by definition of $\delta(t)$ we have

$$\int_{-\infty}^{\infty} \left(\lim_{n\to\infty} nf(nt)\right)g(t)dt = \int_{-\infty}^{\infty} \delta(t)g(t)dt$$

On the other hand, we have

$$\int_{-\infty}^{\infty} f(u)\left(\lim_{n\to\infty} g\left(\frac{u}{n}\right)\right) du = \int_{-\infty}^{\infty} f(u)g(0)du = g(0)\underbrace{\int_{-\infty}^{\infty} f(u)du}_{1} = g(0)$$

$$\text{Hence } \int_{-\infty}^{\infty} \delta(t)g(t)dt = g(0)\checkmark$$