# LECTURE: EXACT EQUATIONS

## 1. Introduction

**Today:** Solve differential equations of the form

$$Pdx + Qdy = 0$$

Here $P$ and $Q$ are functions of $x$ and $y$

Tons of ODE can be put in this form:

**Example 1:**

$$\frac{dy}{dx} = -\left(\frac{2xy + y^2}{x^2 + 2xy}\right)$$

Cross-multiplying, this becomes the same as

$$\left(x^2 + 2xy\right) dy = -\left(2xy + y^2\right) dx$$
$$\underbrace{\left(2xy + y^2\right)}_{P} dx + \underbrace{\left(x^2 + 2xy\right)}_{Q} dy = 0$$

If the notation $Pdx + Qdy$ looks familiar, it's because it is! It's the same notation you used to deal with line integrals in Multivariable Calculus!

## 2. Multivariable Review

Here's a quick review of the multivariable tools that we'll need.

**Recall: (Gradient)**

$$\text{If } f = f(x,y) \text{ then } \nabla f = \langle f_x, f_y \rangle$$

Intuitively $\nabla f$ is the vector of all derivatives of $f$

**Example 2: (Potential Function)**

Let $F(x,y) = \left\langle 2xy + y^2, x^2 + 2xy \right\rangle$ Find $f$ such that $F = \nabla f$

**STEP 1: Check $F$ is conservative:**

**Recall:**

$$F = \langle P, Q \rangle \text{ conservative } \Leftrightarrow P_y = Q_x$$

**Mnemonic: P**e**Y**am = **Q**ui**X**otic

$$\begin{aligned} P_y &= (2xy + y^2)_y = 2x + 2y \\ Q_x &= (x^2 + 2xy)_x = 2x + 2y \end{aligned}$$

Since $P_y = Q_x$ we get that $F$ is conservative ✓

Conservative means $F$ has a potential function, there is $f$ with $F = \nabla f$

**Why?** If $F = \nabla f$ then $\langle P, Q \rangle = \langle f_x, f_y \rangle$ so $P = f_x$ and $Q = f_y$ but by Clairaut, we get

$$f_{xy} = f_{yx} \Rightarrow \left(f_x\right)_y = \left(f_y\right)_x \Rightarrow P_y = Q_x$$

So Conservative $\Rightarrow P_y = Q_x$

The other direction is also true, but harder to show.

**STEP 2: Find $f$**

$$F = \nabla f \Rightarrow \langle P, Q\rangle = \langle f_x, f_y\rangle$$

$$f_x = 2xy + y^2 \Rightarrow f(x,y) = \int 2xy + y^2 dx = x^2y + y^2x + g(y)$$

$$f_y = x^2 + 2xy \Rightarrow f(x,y) = \int x^2 + 2xy dy = x^2y + y^2x + h(x)$$

Here $g$ is any function of $y$ (constant with respect to $x$) and $h$ is any function of $x$

Comparing both equations, we get

$$f(x,y) = x^2y + y^2x$$

## 3. Solving Exact Equations

The method above surprisingly allows us to solve ODEs:

**Example 3:**

Solve $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$

Let $F = \langle P, Q\rangle = \left\langle 2xy + y^2, x^2 + 2xy\right\rangle$

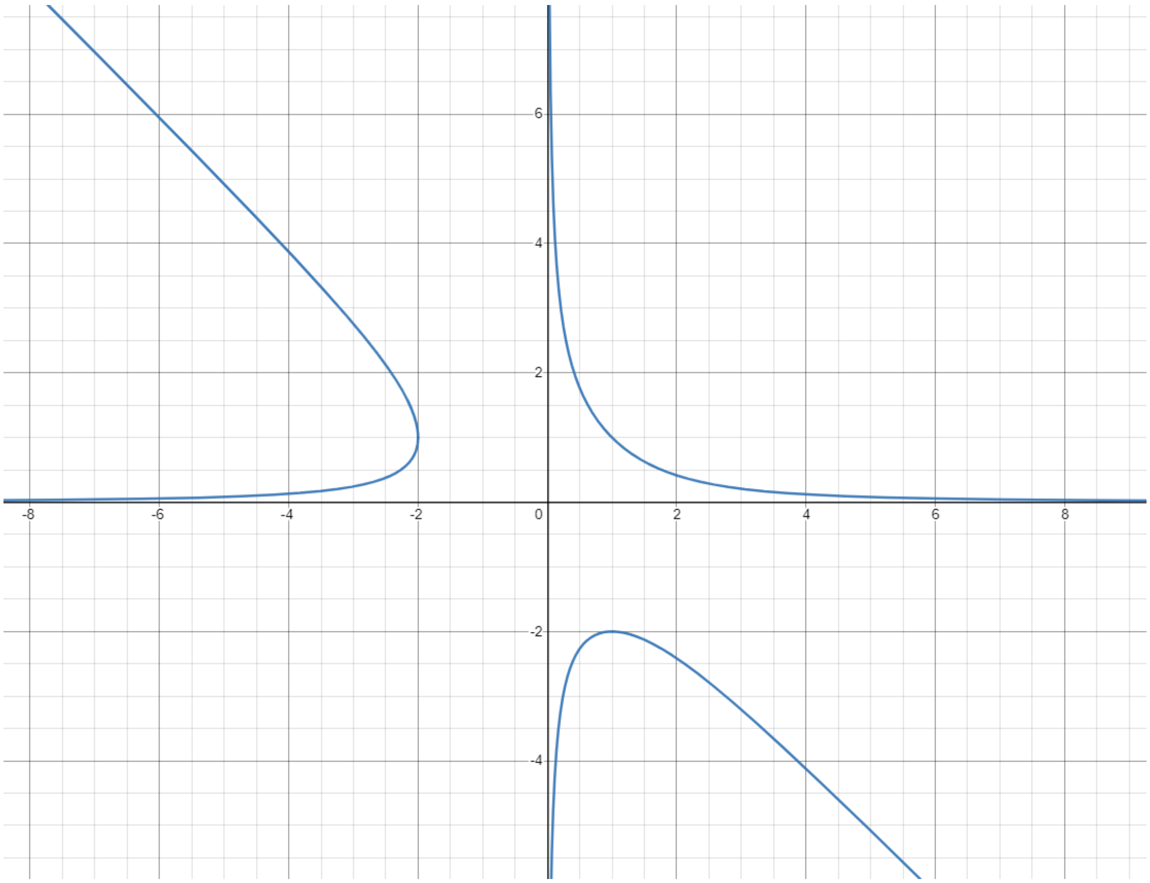
Showed $F = \nabla f$ where $f(x,y) = x^2y + xy^2$

**Fact:**

The solutions are $f(x,y) = C$ where $C$ is any constant

So here the solutions are simply

$$x^2y + xy^2 = C$$




**Definition:**

If $P_y = Q_x$ then $Pdx + Qdy = 0$ is an **exact** ODE

**Example 4:**

$$(3 + 2xy)\, dx + \left(x^2 - 3y^2\right) dy = 0$$

**STEP 1:** Check conservative/exact:

$$\begin{aligned} P_y &= (3 + 2xy)_y = 2x \\ Q_x &= \left(x^2 - 3y^2\right)_x = 2x \\ P_y &= Q_x \checkmark \end{aligned}$$

**STEP 2:** Find $f$

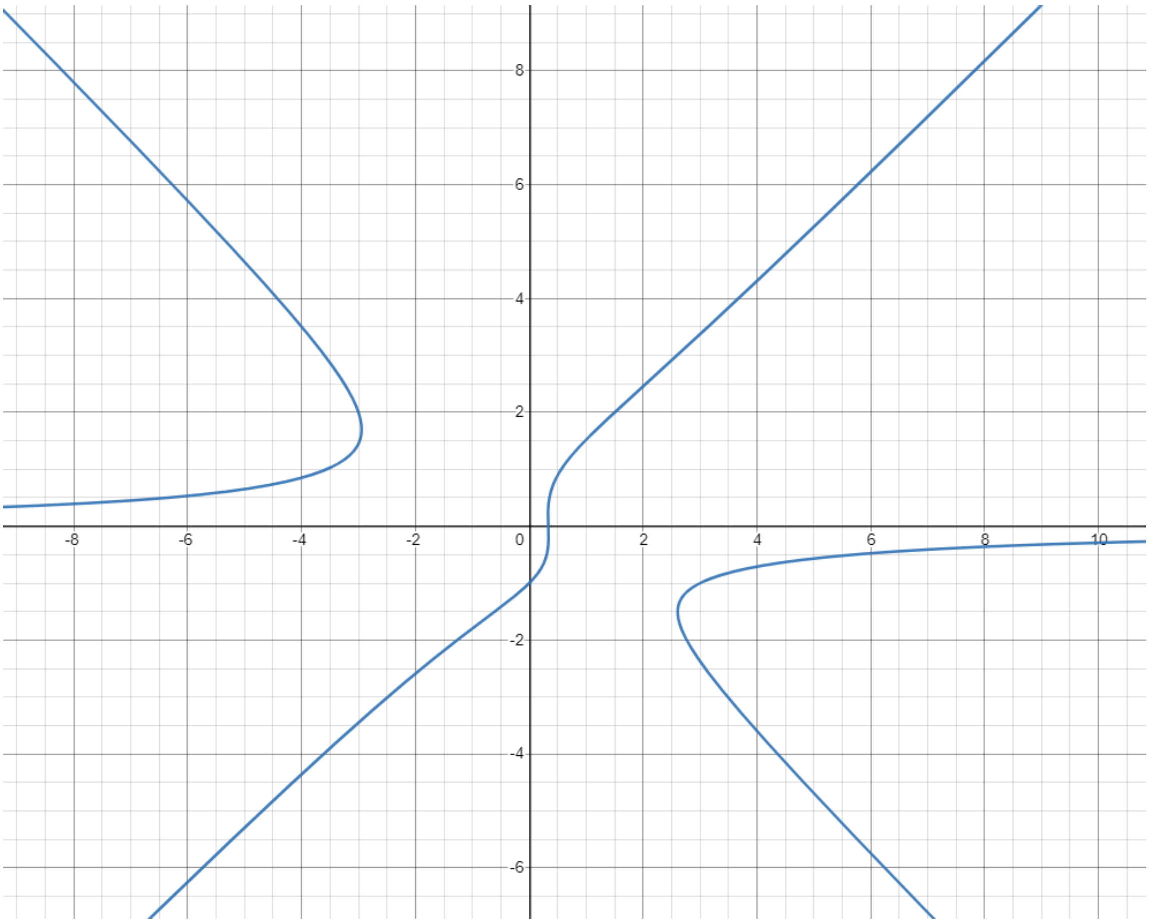
$$\begin{aligned} f_x &= 3 + 2xy \Rightarrow f = \int 3 + 2xy dx = 3x + x^2 y + g(y) \\ f_y &= x^2 - 3y^2 \Rightarrow f = \int x^2 - 3y^2 dy = x^2 y - y^3 + h(x) \end{aligned}$$

$$f(x, y) = 3x + x^2 y - y^3$$

(Don't double-count the $x^2y$)

**STEP 3:** Solution

$$3x + x^2 y - y^3 = C$$




**Example 5: (extra practice)**

$$\begin{cases} y\cos(x) + 2xe^y + \left(\sin(x) + x^2e^y - 1\right) y' = 0 \\ y(0) = 1 \end{cases}$$

**Note:** Since $y' = \frac{dy}{dx}$, this is the same as saying

$$(y\cos(x) + 2xe^y)\, dx + \left(\sin(x) + x^2e^y - 1\right) dy = 0$$

**STEP 1:** Check $F$ conservative (exact)

$$\begin{aligned} P_y &= (y\cos(x) + 2xe^y)_y = \cos(x) + 2xe^y \\ Q_x &= \left(\sin(x) + x^2e^y - 1\right)_x = \cos(x) + 2xe^y \\ P_y &= Q_x \checkmark \end{aligned}$$

**STEP 2:** Find $f$

$$\begin{aligned} f_x &= y\cos(x) + 2xe^y \Rightarrow f = \int y\cos(x) + 2xe^y dx = y\sin(x) + x^2e^y + g(y) \\ f_y &= \sin(x) + x^2e^y - 1 \Rightarrow f = \int \sin(x) + x^2e^y - 1dy = y\sin(x) + x^2e^y - y + h(x) \end{aligned}$$

$$f(x,y) = y\sin(x) + x^2e^y - y$$

**STEP 3:** Solution

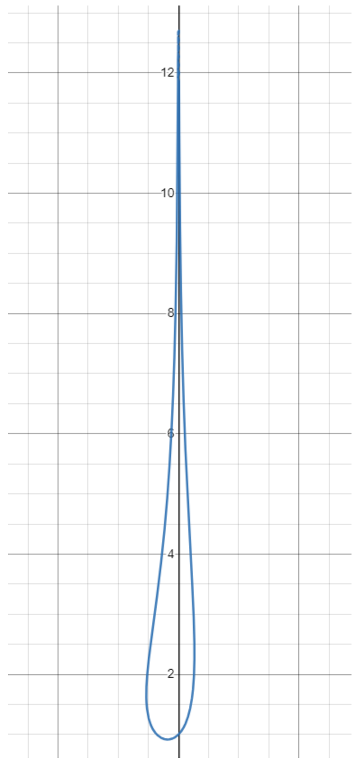
$$y\sin(x) + x^2e^y - y = C$$

**STEP 4: Initial Condition:** $y(0) = 1$.

Plug in $x = 0$ and $y = 1$ in the equation above:

$$1\sin(0) + 0^2e^0 - 1 = C \Rightarrow C = -1$$

**Answer:** $y\sin(x) + x^2e^y - y = -1$



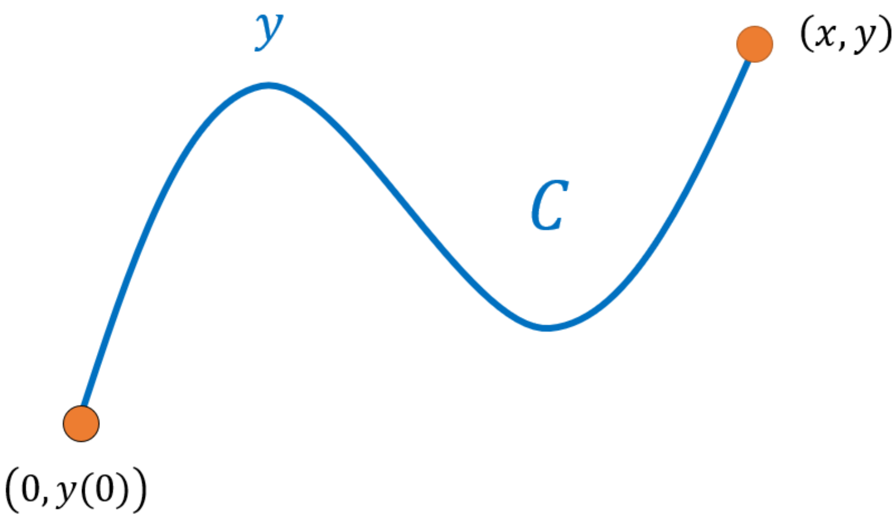

**Extra Practice:** For an extra example, check out the following video:

**Video:** Exact Equations

## 4. Why this works

Here is why this method works. It is a beautiful application of the FTC for line integrals.

Let $C$ be the solution curve from $(0, y(0))$ (initial condition) to a given point $(x, y)$




Start with $Pdx + Qdy = 0$ and integrate over $C$:

$$\int_C Pdx + Qdy = \int_C 0 = 0$$

On the other hand, if you let $F = \langle P, Q \rangle = \nabla f$ then

$$0 = \int_C Pdx + Qdy = \int_C F \cdot dr = \int_C \nabla f \cdot dr \overset{\text{FTC}}{=} f(\text{end}) - f(\text{start})$$
$$= f(x, y) - f(0, y(0))$$

$$\text{Hence } f(x, y) - \underbrace{f(0, y(0))}_{\text{Constant}} = 0 \Rightarrow f(x, y) = \text{ Constant}$$

Which is what we wanted to show!

**Note:** To show that $y$ actually solves the ODE, you start with

$$f(x, y) = C$$

And differentiate this with respect to $x$, using the multivariable chain rule and keeping in mind that $y = y(x)$

$$\begin{aligned}
\frac{d}{dx} f(x, y(x)) =& \frac{d}{dx} C \\
f_x(x, y)\left(\frac{dx}{dx}\right) + f_y(x, y)\left(\frac{dy}{dx}\right) =& 0 \\
P + Q\left(\frac{dy}{dx}\right) =& 0 \qquad \text{Since } \langle f_x, f_y \rangle = \nabla f = \langle P, Q \rangle \\
Pdx + Qdy =& 0 \qquad \text{Multiply by } dx
\end{aligned}$$

Hence $y$ indeed solves the ODE $Pdx + Qdy = 0$ ✓

## 5. Non-Exact Equations

**Example 6:**

$$\left(3xy + y^2\right) dx + \left(x^2 + xy\right) dy = 0$$

**STEP 1:** Check exact

$$\begin{aligned}
P_y = \left(3xy + y^2\right)_y = 3x + 2y \\
Q_x = \left(x^2 + xy\right)_x = 2x + y
\end{aligned}$$

**OH NO!!!** $P_y \neq Q_x$ so the equation is not exact, and there's not much we can do

**BUUUUUUT** sometimes we can multiply the (inexact) equation by an integrating factor to make it exact.

**Trick:** Multiply the ODE by $x$ (this will be given):

$$\begin{aligned}
x\left(3xy + y^2\right) dx + x\left(x^2 + xy\right) dy =& x(0) \\
\left(3x^2y + xy^2\right) dx + \left(x^3 + x^2y\right) dy =& 0
\end{aligned}$$

**STEP 1:** (again) Check exact

$$\begin{aligned} P_y &= \left(3x^2y + xy^2\right)_y = 3x^2 + x(2y) = 3x^2 + 2xy \\ Q_x &= \left(x^3 + x^2y\right)_x = 3x^2 + 2xy \\ P_y &= Q_x \checkmark \end{aligned}$$

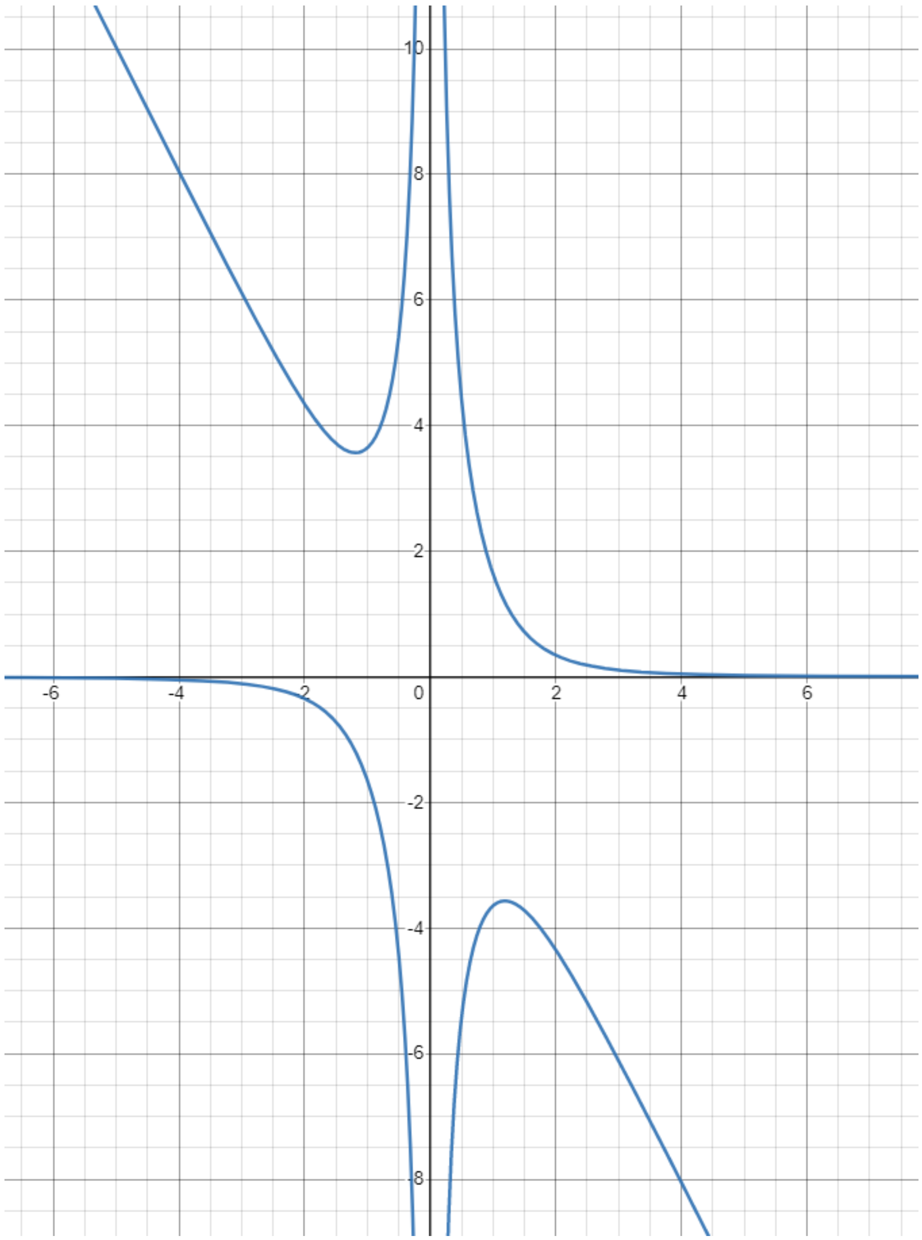
**STEP 2:** Find $f$

$$\begin{aligned} f_x &= 3x^2y + xy^2 \Rightarrow f = \int 3x^2y + xy^2 dx = x^3y + \frac{1}{2}x^2y^2 + g(y) \\ f_y &= x^3 + x^2y \Rightarrow f = \int x^3 + x^2y dy = x^3y + \frac{1}{2}x^2y^2 + h(x) \end{aligned}$$

$$f(x, y) = x^3y + \frac{1}{2}x^2y^2$$

**STEP 3: Solution**

$$x^3y + \frac{1}{2}x^2y^2 = C$$




**Aside:** How to obtain that integrating factor $x$?

Suppose our integrating factor is $g(x, y)$, then multiplying by $g$, we get

$$
\begin{aligned}
Pdx + Qdy =& 0 \\
g\,(Pdx + Qdy) =& g0 \\
(Pg)dx + (Qg)dy =& 0
\end{aligned}
$$

Since we want the above to be exact, we require

$$(Pg)_y = (Qg)_x$$

This gives a *partial differential equation* for $g$ which, is hard to solve in practice. You can simplify this for example by requiring $g$ to be a function of $x$, $g = g(x)$, but then you're not guaranteed to have a solution