APMA 0350 – MIDTERM 1 – SOLUTIONS

1. **Integrating Factors**

STEP 1: Standard Form

$$y' - \left(\frac{3}{t}\right) y = t^4$$

STEP 2: Integrating Factor

$$e^{\int -\frac{3}{t} dt} = e^{-3\ln(t)} = \left(e^{\ln(t)}\right)^{-3} = t^{-3}$$

Here we used $t > 0$ so $|t| = t$

STEP 3: Multiply by t^{-3}

$$\begin{aligned} t^{-3}y' + t^{-3}\left(\frac{3}{t}\right) y =& t^{-3}\left(t^4\right) \\ \left(t^{-3}y\right)' =& t \\ t^{-3}y =& \int t dt = \frac{1}{2}t^2 + C \\ y =& t^3\left(\frac{t^2}{2} + C\right) \\ y =& \frac{t^5}{2} + Ct^3 \end{aligned}$$

STEP 4: Initial Condition

$$\begin{aligned} y(1) =& \frac{5}{2} \\ \frac{1^5}{2} + C(1)^3 =& \frac{5}{2} \\ \frac{1}{2} + C =& \frac{5}{2} \\ C =& 2 \end{aligned}$$

STEP 5: Solution

$$y = \frac{1}{2}t^5 + 2t^3$$

2. **Exact Equations**

STEP 1: Rewrite the ODE

$$\left(x^2 - \cos(y)\right) dy = (\sin(x) - 2xy)\, dx$$
$$(\sin(x) - 2xy)\, dx + \left(\cos(y) - x^2\right) dy = 0$$

STEP 2: Check exact

$$P_y = (\sin(x) - 2xy)_y = -2x$$
$$Q_x = \left(\cos(y) - x^2\right)_x = -2x\checkmark$$

STEP 3:

$$f_x = \sin(x) - 2xy \Rightarrow f = \int \sin(x) - 2xydx = -\cos(x) - x^2y + g(y)$$
$$f_y = \cos(y) - x^2 \Rightarrow f = \int \cos(y) - x^2dy = \sin(y) - x^2y + h(x)$$

Comparing, we get $g(y) = \sin(y)$ and $h(x) = -\cos(x)$ and therefore

$$f(x, y) = -\cos(x) - x^2y + \sin(y)$$

STEP 4: General Solution:

$$-\cos(x) - x^2y + \sin(y) = C$$

STEP 5: Initial Condition

$$y\left(\frac{\pi}{2}\right) = 0$$
$$-\cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^2 0 + \sin(0) = C$$
$$0 = C$$
$$C = 0$$

$$-\cos(x) - x^2y + \sin(y) = 0$$

3. STEP 1: Auxiliary Equation

$$\begin{aligned} r^2+4r+20 &= 0 \\ (r+2)^2-2^2+20 &= 0 \\ (r+2)^2+16 &= 0 \\ (r+2)^2 &= -16 \\ r+2 &= \pm 4i \\ r &= -2\pm 4i \end{aligned}$$

$$y = Ae^{-2t}\cos(4t) + Be^{-2t}\sin(4t)$$

STEP 2: Initial Condition

$$\begin{aligned} y(0) &= 2 \\ Ae^0\cos(0)+Be^0\sin(0) &= 2 \\ A &= 2 \end{aligned}$$

$$y = 2e^{-2t}\cos(4t) + Be^{-2t}\sin(4t)$$

$$y' = -4e^{-2t}\cos(4t)+2e^{-2t}(-4\sin(4t))-2Be^{-2t}\sin(4t)+Be^{-2t}(4\cos(4t))$$

$$\begin{aligned} y'(0) &= -8 \\ -4e^0\cos(0)-8e^0\sin(0)-2Be^0\sin(0)+4Be^0\cos(0) &= -8 \\ -4+4B &= -8 \\ 4B &= -8+4=-4 \\ B &= -1 \end{aligned}$$

STEP 3: Answer:

$$y = 2e^{-2t}\cos(4t) - e^{-2t}\sin(4t)$$

4. **STEP 1: Equilibrium Solution**

$$y' = 0 \Rightarrow -3\,(y+2)^3\, y\,(y-2)^2\,(y-4) = 0$$

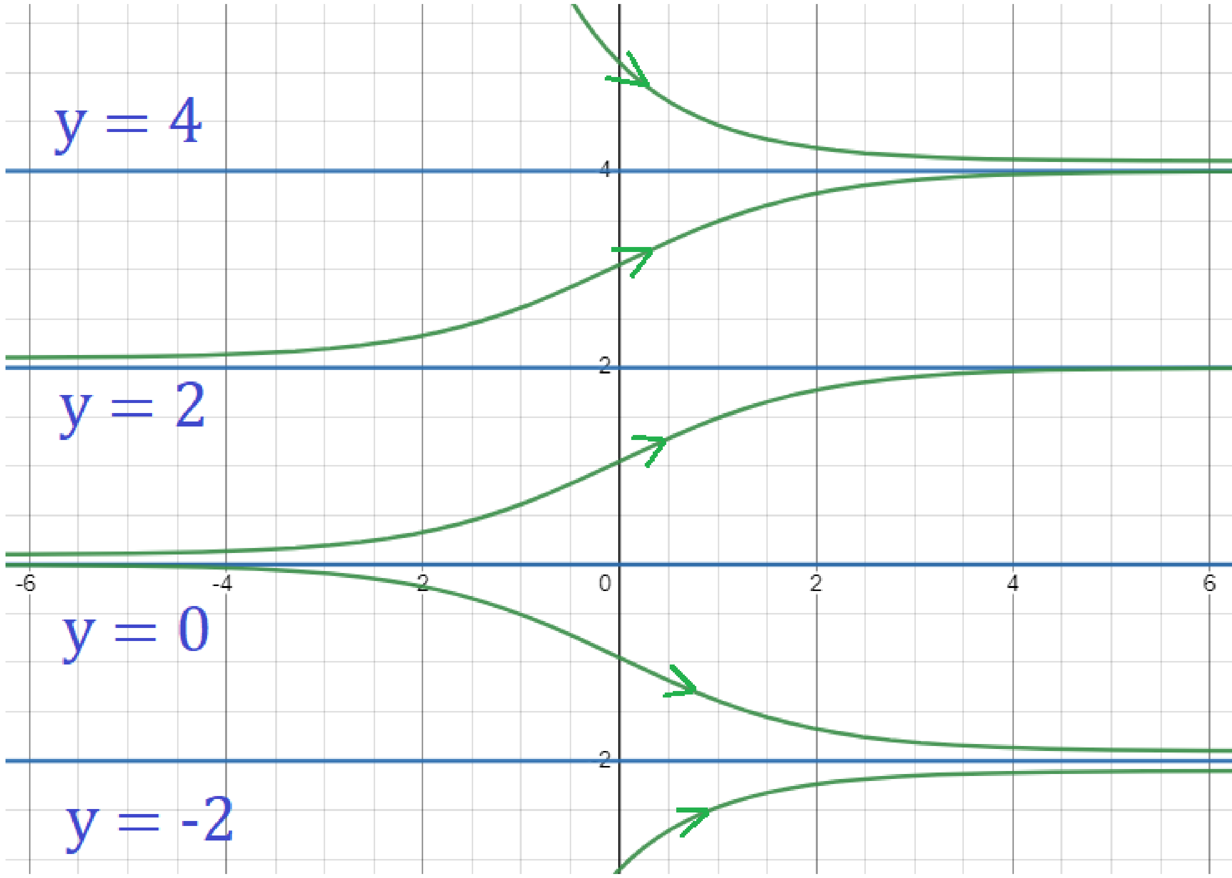
Which gives the equilibrium solutions

$$y = -2 \text{ and } y = 0 \text{ and } y = 2 \text{ and } y = 4$$

STEP 2: Sign Table

y	$-\infty$	-2		0		2		4	∞
$-3(y+2)^3$	+	0	−		−		−		−
y	−		−	0	+		+		+
$(y-2)^2$	+		+		+	0	+		+
y-4	−		−		−		−	0	+
y'	+	0	−	0	+	0	+	0	−
y	↗		↘		↗		↗		↘

STEP 3: Sample solutions (optional)



STEP 4: Answer: From the picture above, it looks like

$$y = -2 \text{ is stable}$$
$$y = 0 \text{ is unstable}$$
$$y = 2 \text{ is bistable}$$
$$y = 4 \text{ is stable}$$

5. **STEP 1:** Differential Equation

$$P'(t) = 3P(t) - 15$$

Aside: Although you don't need to do it, one way to derive the ODE is to notice that

$$\text{Change in } P \; = P(t+h) - P(t) = 3P(t)h - 15h$$

Where the $3P(t)h$ is due to the offspring (depends on the number of rabbits present) and the $-15h$ is due to the number of rabbits eaten (does not depend on the existing population)

Dividing by h and letting $h \to 0$ gives the differential equation

STEP 2: Solve the ODE

$$\begin{cases} P'(t) = 3P(t) - 15 \\ P(0) = 100 \end{cases}$$

We'll use the method of integrating factors

$$P'(t) - 3P(t) = -15$$

The integrating factor here is e^{-3t}

$$\begin{aligned}
e^{-3t}P'(t) - 3e^{-3t}P(t) =& -15e^{-3t} \\
\left(e^{-3t}P(t)\right)' =& -15e^{-3t} \\
e^{-3t}P(t) =& \int -15e^{-3t}dt = -15\left(\frac{e^{-3t}}{-3}\right) + C \\
e^{-3t}P(t) =& 5e^{-3t} + C \\
P(t) =& e^{3t}\left(5e^{-3t} + C\right) \\
P(t) =& 5 + Ce^{3t}
\end{aligned}$$

$$\begin{aligned}
P(0) =& 100 \\
5 + Ce^{0} =& 100 \\
5 + C =& 100 \\
C =& 95
\end{aligned}$$

STEP 4: Answer

$$P(t) = 5 + 95e^{3t}$$