APMA 1650 – Homework 6 Answer Key

1. You are again the quality control manager for the Acme Widget Company. You have just launched a new line of MiniWidgets. Your MiniWidgets are produced by a MiniWidget machine. The MiniWidgets produced by the machine have masses which are normally distributed with a standard deviation of 0.2 grams. The machine can be adjusted so that the MiniWidgets it produces have an average mass of μ grams. What setting for μ should you use so that the masses of the MiniWidgets will exceed 10 grams at most 2% of the time?

   Let X be the mass of the MiniWidgets. Then, converting to the standard normal random variable:

$$\begin{aligned} 0.02 &= \mathbb{P}(X > 10) \\ 0.98 &= \mathbb{P}(X \leq 10) \\ &= \mathbb{P}\left(Z \leq \frac{10-\mu}{0.2}\right) \end{aligned}$$

   Looking at the Z table for a probability of 0.98 we find that $z = 2.06$. Setting the quantity above to 2.06, we find:

$$\begin{aligned} \frac{10-\mu}{0.2} &= 2.06 \\ 10-\mu &= (2.06)(0.2) = 0.412 \\ \mu &= 10 - 0.412 = 9.588 \end{aligned}$$

2. Let X_1 and X_2 be two independent geometric random variables, both with parameter p. Find a nice, closed-form formula for

$$\mathbb{P}(X_1 = i | X_1 + X_2 = n)$$

   Your formula should not involve a sum. Hint: use the definition of conditional probability.

   By the definition of conditional probability,

$$\begin{aligned} \mathbb{P}(X_1 = i | X_1 + X_2 = n) &= \frac{\mathbb{P}(X_1 = i, X_1 + X_2 = n)}{\mathbb{P}(X_1 + X_2 = n)} \\ &= \frac{\mathbb{P}(X_1 = i, X_2 = n - i)}{\mathbb{P}(X_1 + X_2 = n)} \\ &= \frac{\mathbb{P}(X_1 = i)\mathbb{P}(X_2 = n - i)}{\mathbb{P}(X_1 + X_2 = n)} \end{aligned}$$

where in the last line we have used the independence of X_1 and X_2. We can compute all these probabilities using the geometric pmf.

$$\begin{aligned} \mathbb{P}(X_1 = i) &= (1-p)^{i-1}p \\ \mathbb{P}(X_1 = n-i) &= (1-p)^{n-i-1}p \end{aligned}$$

For the denominator, note that if $X_1+X_2 = n$, there are $(n-1)$ mutually exclusive ways this can occur: $(X_1 = 1, X_2 = n-1), (X_1 = 2, X_2 = n-2), \dots, (X_1 = n-1, X_2 = 1)$. Thus we have:

$$\begin{aligned} \mathbb{P}(X_1 + X_2 = n) &= \sum_{k=1}^{n-1} \mathbb{P}(X_1 = k, X_2 = n-k) \\ &= \sum_{k=1}^{n-1} \mathbb{P}(X_1 = k)\mathbb{P}(X_2 = n-k) \\ &= \sum_{k=1}^{n-1} (1-p)^{k-1}p(1-p)^{n-k-1}p \\ &= \sum_{k=1}^{n-1} (1-p)^{n-2}p^2 \\ &= (n-1)(1-p)^{n-2}p^2 \end{aligned}$$

since all terms in the sum are the same. Plugging all of this in, we get:

$$\begin{aligned} \mathbb{P}(X_1 = i | X_1 + X_2 = n) &= \frac{(1-p)^{i-1}p(1-p)^{n-i-1}p}{(n-1)(1-p)^{n-2}p^2} \\ &= \frac{(1-p)^{n-2}p^2}{(n-1)(1-p)^{n-2}p^2} \\ &= \frac{1}{n-1} \end{aligned}$$

3. Suppose that the random variables Y_1 and Y_2 have joint density function:

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show this is a valid joint probability density.

The region of interest is a square with opposite corners $(0,0)$ and $(1,1)$, so the limits on both y_1 and y_2 will be from 0 to 1. As always, draw the region! It does not matter if we integrate with respect to y_1 or y_2 first, so will start with y_1.

$$\int_0^1 \int_0^1 (y_1 + y_2) dy_1 dy_2 = \int_0^1 \left(\frac{y_1^2}{2} + y_2 y_1 \right) \Big|_0^1 dy_2$$
$$= \int_0^1 \left(\frac{1}{2} + y_2 \right) dy_2$$
$$= \left(\frac{1}{2} y_2 + \frac{y_2^2}{2} \right)_0^1$$
$$= \frac{1}{2} + \frac{1}{2} = 1$$

(b) Find the marginal densities for Y_1 and Y_2.

To find the marginal density of Y_1, we integrate with respect to y_2 to get rid of it.

$$f_1(y_1) = \int_0^1 (y_1 + y_2) dy_2$$
$$= \left(y_1 y_2 + \frac{y_2^2}{2} \right)_0^1$$
$$= y_1 + \frac{1}{2}$$

Thus the marginal density of Y_1 is

$$f_1(y_1) = \begin{cases} y_1 + \frac{1}{2} & 0 \leq y_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

By symmetry, the marginal density of Y_2 is

$$f_2(y_2) = \begin{cases} y_2 + \frac{1}{2} & 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the probability that $Y_1 < 1/2$ and $Y_2 > 1/2$.

Here we integrate the joint density with respect to y_1 from 0 to 1/2 and with respect to y_2 from 1/2 to 1.

$$\begin{aligned}
\mathbb{P}(Y_1 < 1/2, Y_2 > 1/2) &= \int_{1/2}^{1}\int_{0}^{1/2}(y_1+y_2)dy_1dy_2 \\
&= \int_{1/2}^{1}\left(\frac{y_1^2}{2}+y_2y_1\right)\Big|_0^{1/2}dy_2 \\
&= \int_{1/2}^{1}\left(\frac{1}{8}+\frac{1}{2}y_1\right)dy_1 \\
&= \left(\frac{1}{8}y_1+\frac{1}{4}y_1^2\right)\Big|_{1/2}^{1} \\
&= \left(\frac{1}{8}+\frac{1}{4}\right)-\left(\frac{1}{16}+\frac{1}{16}\right) \qquad = \frac{1}{4}
\end{aligned}$$

(d) Find the conditional density for Y_1 given $Y_2 = y_2$.
The conditional density is the joint density divided by the marginal density.

$$f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{y_1+y_2}{y_2+1/2}$$

We need to put bounds on this, but they are still from 0 to 1 in this case.

$$f(y_1|y_2) = \begin{cases} \frac{y_1+y_2}{y_2+1/2} & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(e) Find the probability that $Y_1 < 1/2$ given $Y_2 = 1/2$.
To find this probability, we work with the conditional density from the previous part. Plugging in $y_2 = 1/2$ into the conditional density, we get $f(y_1|Y_2 1/2) = y_1 + 1/2$. Integrating this from 0 to 1/2 we get:

$$\begin{aligned}
\mathbb{P}(Y_1 < 1/2|Y_2 = 1/2) &= \int_0^{1/2}(y_1+1/2)dy_1 \\
&= \left(\frac{y_1^2}{2}+\frac{y_1}{2}\right)\Big|_0^{1/2} \\
&= \frac{1}{8}+\frac{1}{4} = \frac{3}{8}
\end{aligned}$$

4. Suppose that the random variables Y_1 and Y_2 have joint density function:

$$f(y_1,y_2) = \begin{cases} c(1-y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid joint probability density.

The first step is to draw the region so that you have the correct limits of integration. This region is drawn in my course notes, so you can refer to the drawing there. We will integrate in the y_1 direction first since that lets us use 0 as the lower limit of integration twice.

$$\begin{aligned}
1 &= \int_0^1 \int_0^{y_2} c(1-y_2)dy_1dy_2 \\
&= c\int_0^1 (y_1 - y_2y_1)\Big|_0^{y_2} dy_2 \\
&= c\int_0^1 (y_2 - y_2^2)dy_2 \\
&= c\left(\frac{y_2^2}{2} - \frac{y_2^3}{3}\right)\Big|_0^1 \\
&= c\left(\frac{1}{2} - \frac{1}{3}\right) \\
&= \frac{c}{6}
\end{aligned}$$

From this we conclude that $c = 6$.

(b) Find the marginal densities for Y_1 and Y_2.

To do this we integrate with respect to the other variable. Do not forget to look at the picture to make sure you have the correct limits!

$$\begin{aligned}
f_1(y_1) &= \int_{y_1}^1 6(1-y_2)dy_2 \\
&= (6y_2 - 3y_2^2)\Big|_{y_1}^1 \\
&= (6-3) - (6y_1 - 3y_1^2) \\
&= 3 - 6y_1 + 3y_1^2
\end{aligned}$$

The marginal density of Y_1 is:

$$f_1(y_1) = \begin{cases} 3 - 6y_1 + 3y_1^2 & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
f_1(y_1) &= \int_0^{y_2} 6(1-y_2)dy_1 \\
&= 6(y_1 - y_2y_1)\Big|_0^{y_2} \\
&= 6(y_2 - y_2^2)
\end{aligned}$$

The marginal density of Y_2 is:

$$f_2(y_2) = \begin{cases} 6(y_2 - y_2^2) & 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the conditional density for Y_1 given $Y_2 = y_2$.

For the conditional density, we divide the joint density by the marginal density.

$$f(y_1|y_2) = \frac{6(1-y_2)}{6(y_2 - y_2^2)} = \frac{(1-y_2)}{(y_2(1-y_2)} = \frac{1}{y_2}$$

We need to put bounds on this. Looking at the picture, we see that if $Y_2 = y_2$, Y_1 can only range from 0 to y_2. Thus we have:

$$f(y_1|y_2) = \begin{cases} \frac{1}{y_2} & 0 \leq y_1 \leq y_2 \\ 0 & \text{otherwise} \end{cases}$$

Note that we also have to assume that $y_2 \neq 0$.

(d) Find the expected values $\mathbb{E}(Y_1)$ and $\mathbb{E}(Y_2)$.

To find these expected values, use the marginal densities with the formula for the expected value of a continuous random variable.

$$\begin{aligned} \mathbb{E}(Y_1) &= \int_0^1 y_1(3 - 6y_1 + 3y_1^2)dy_1 \\ &= 3\int_0^1 (y_1 - 2y_1^2 + y_1^3)dy_1 \\ &= 3\left(\frac{y_1^2}{2} - 2\frac{y_1^3}{3} + \frac{y_1^4}{4}\right)\Big|_0^1 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbb{E}(Y_2) &= \int_0^1 y_2 6(y_2 - y_2^2)dy_2 \\ &= 6\int_0^1 (y_2^2 - y_2^3)dy_2 \\ &= 6\left(\frac{y_2^3}{3} - \frac{y_2^4}{4}\right) \\ &= 6\left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \frac{1}{2} \end{aligned}$$

5. Suppose that the random variables Y_1 and Y_2 have joint density function:

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)} & y_1 > 0, y_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal densities for Y_1 and Y_2. What kind of random variables are Y_1 and Y_2?

For the marginal density of Y_1, we integrate with respect to y_2.

$$\begin{aligned} f_1(y_1) &= \int_0^\infty e^{-(y_1+y_2)} dy_2 \\ &= \int_0^\infty e^{-y_1} e^{-y_2} dy_2 \\ &= e^{-y_1} \lim_{t\to\infty} (-e^{-y_2})_0^t \\ &= e^{-y_1}(1 - \lim_{t\to\infty} e^{-t}) \\ &= e^{-y_1} \end{aligned}$$

With appropriate bounds, we have marginal density

$$f_1(y_1) = \begin{cases} e^{-y_1} & y_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

By symmetry,

$$f_2(y_2) = \begin{cases} e^{-y_2} & y_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

We recognize both of these densities as exponential random variables with parameter $\lambda = 1$.

(b) Find the conditional density of Y_1 given that $Y_2 = y_2$ for $y_2 > 0$.

Dividing the joint density by the marginal density,

$$f(y_1|y_2) = \frac{e^{-(y_1+y_2)}}{e^{-y_2}} = \frac{e^{-y_1}e^{-y_2}}{e^{-y_2}} = e^{-y_1}$$

So with bounds, this is

$$f_1(y_1|y_2) = \begin{cases} e^{-y_1} & y_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

This is the same as the marginal density of Y_1.

(c) Are Y_1 and Y_2 independent? Justify your answer. Since the conditional density of Y_1 given $Y_2 = y_2$ is the same as the marginal density of y_1, the random variables Y_1 and Y_2 are independent. Alternatively, since the joint density is the product of the two marginal densities, the two random variables are independent.