LECTURE: JOINT EXPECTATION

1. Another Example

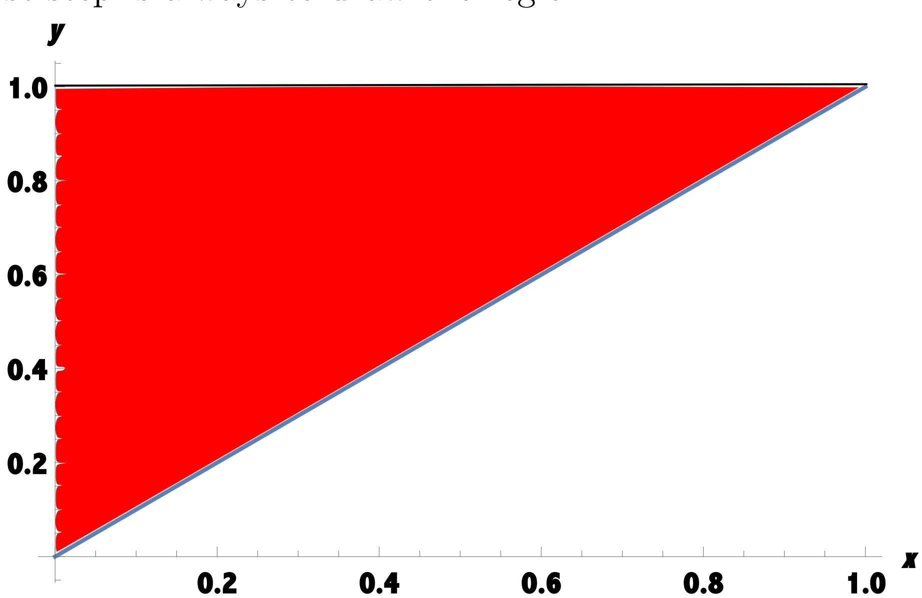
Example 1:

Let X and Y be random variables with joint density $f(x,y)$ where

$$f(x,y) = \begin{cases} cx & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a valid joint probability density function

The first step is always to draw the region.



$$1 = \int_0^1 \int_0^y cx dx dy = c \int_0^1 \left[\frac{x^2}{2} \right]_{x=0}^{x=y} dy = c \int_0^1 \frac{y^2}{2} dy = c \left[\frac{y^3}{6} \right]_0^1 = \frac{c}{6}$$

Therefore $c = 6$

(b) Find the marginal densities for X and Y

For the marginal density of X we first integrate out y.

$$f_X(x) = \int_x^1 6x dy = [6xy]_{y=x}^{y=1} = 6x(1-x)$$

x can freely range from 0 to 1, so the marginal density of X is:

$$f_X(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For the marginal density of Y, we first integrate out x:

$$f_Y(y) = \int_0^y 6x dx = \left[3x^2\right]_{x=0}^{x=y} = 3y^2$$

y can freely range from 0 to 1, so the marginal density of Y is:

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) What is the expected value of X?

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x 6x(1-x) dx = 6 \int_0^1 (x^2 - x^3) dx$$
$$= 6 \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 6 \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{2}$$

Similarly we can find the expected value of Y using the marginal density for Y.

(d) What is the conditional density for X given $(Y = y)$?

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6x}{3y^2} = \frac{2x}{y^2}$$

Given that $Y = y$, x can range from 0 to y, thus the cond. density is:

$$f(x|y) = \begin{cases} \frac{2x}{y^2} & 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(e) What is the expected value of X given $(Y = y)$?

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f(x|y) dx = \int_0^y x \frac{2x}{y^2} dx = \int_0^y \frac{2x^2}{y^2} dx = \left[\frac{2x^3}{3y^2} \right]_{x=0}^{x=y} = \frac{2y}{3}$$

Fun Fact: There is in fact an analog of Bayes' Formula, provided you replace sums with integrals!

2. Joint Uniform Distribution

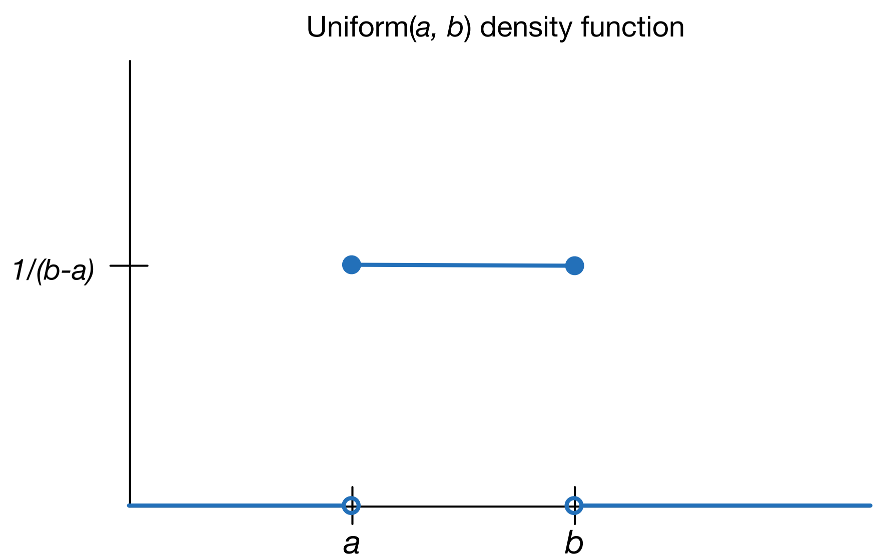
As an example of a joint distribution, we will consider the bivariate uniform distribution.

Recall:

Y has uniform distribution on $[a, b]$ if the density of Y is

$$f(y) = \begin{cases} \frac{1}{b-a} & \text{if } a \le y \le b \\ 0 & \text{otherwise} \end{cases}$$

We write $Y \sim$ Unif (a, b)



Here $b - a$ is the length of $[a, b]$

For the two-dimensional uniform distribution, it's the same thing, except that we replace length with areas:

Definition:

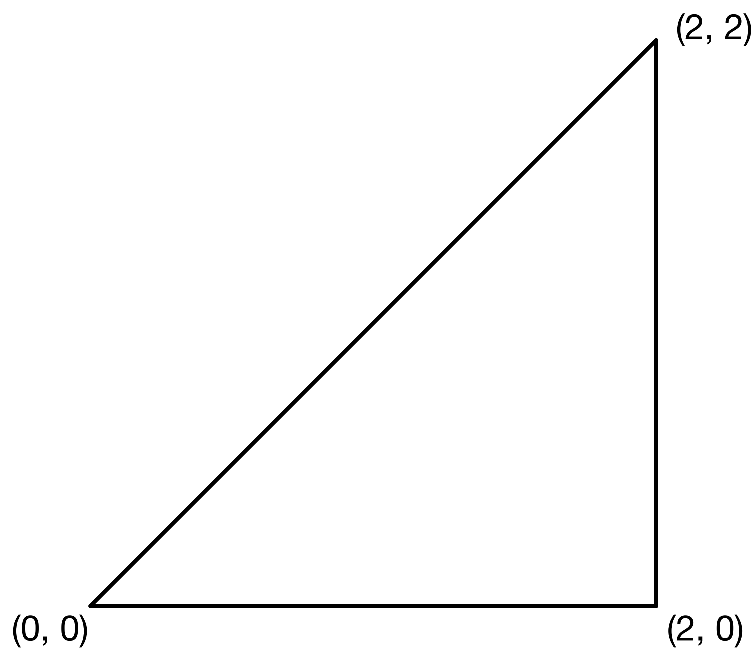
(Y_1, Y_2) have a **joint uniform distribution** on a set A if the joint pdf of (Y_1, Y_2) is

$$f(y_1, y_2) = \begin{cases} \frac{1}{\text{Area }(A)} & \text{if } (y_1, y_2) \in A \\ 0 & \text{otherwise} \end{cases}$$

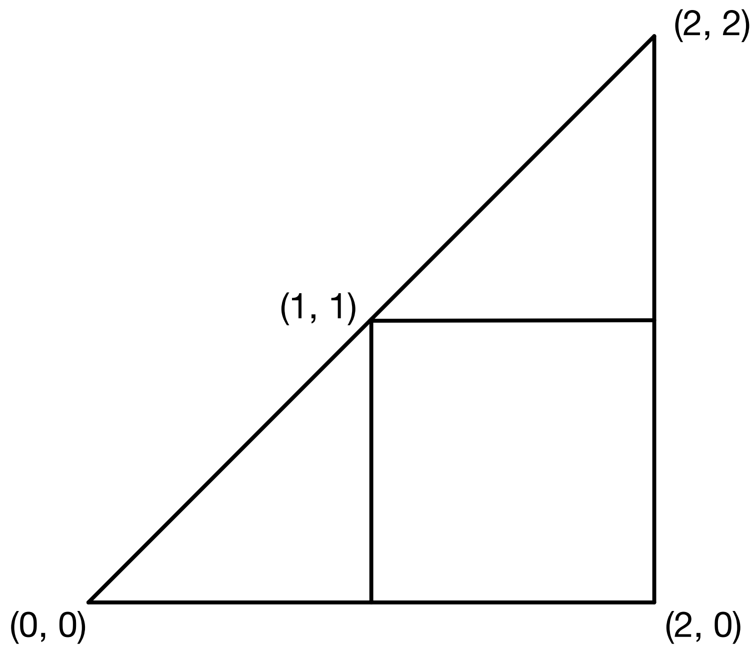
You can check that, in this way, f is a valid probability density, that is, it's ≥ 0 and its total area is 1

Example 2:

Let X and Y have a joint uniform distribution on the equilateral right triangle A below with sides of length 2.



(a) What is the probability that the pair (X, Y) lies within the small square below, with corners $(1, 1)$ and $(2, 0)$?



The small square is half the area of the right triangle, so, since this is the uniform distribution, the probability that (X, Y) lies within the small square is $1/2$.

(b) What is the joint probability density of (X, Y)?

Since the area of the triangle is 2 and, looking at the picture above, we see that $y \geq 0$, $y \leq x$, and $x \leq 2$. Thus the joint density is:

$$f(x, y) = \begin{cases} \frac{1}{2} & 0 \leq y \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) What are the marginal densities of X and Y?

As always, we refer to the picture of the region to get the correct limits of integration.

For the marginal density of X we integrate over y:

$$f_X(x) = \int_0^x \frac{1}{2} dy = \frac{1}{2} [y]_{y=0}^{y=x} = \frac{x}{2}$$

With the correct bounds, the marginal density of X is:

$$f_X(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

For the marginal density of Y we integrate over x:

$$f_Y(y) = \int_y^2 \frac{1}{2} dx = \frac{1}{2} [x]_{x=y}^{1} = \frac{2-y}{2}$$

With the correct bounds, the marginal density of Y is:

$$f_Y(y) = \begin{cases} \frac{2-y}{2} & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

3. Joint Expectation

Recall that in the 1D case, we defined the expected value of a function $g(y)$ of a random variable Y to be:

Recall:

If Y is a discrete random variable with pmf $p(y)$ then

$$E[g(Y)] = \sum_{\text{all } y} g(y)p(y)$$

If Y is a continuous random variable with density $f(y)$ then

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)f(y)dy$$

The expected value of a function of two random variables is defined in the same way, except we use the joint pmf or joint density.

Definition:

If Y_1 and Y_2 be two discrete random variables with joint pmf $p(y_1, y_2)$ then

$$E[g(Y_1, Y_2)] = \sum_{\text{all } y_1} \sum_{\text{all } y_2} g(y_1, y_2)p(y_1, y_2)$$

Definition:

If Y_1 and Y_2 be two cont. random variables with joint density $f(y_1, y_2)$ then

$$E[g(Y_1, Y_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1, y_2)f(y_1, y_2)dy_1dy_2$$