APMA 1650 – MIDTERM 1 – SOLUTIONS

1.

$$\underline{R}\,\underline{E}\,\underline{\text{ Letter }}\,\underline{4}\,\underline{5}\,\underline{\text{ Number }}$$

STEP 1: There are 26 choices for the Letter

STEP 2: For the numbers, there are $3 \times 2 = 6$ ways of placing the 4 and 5 to get a combination like

$$\underline{1}\,\underline{2}\,\underline{\text{ Number }}$$

First assume that the other number is neither 4 and 5, then there are $6 \times 8 = 48$ choices with 4, 5, and another number.

Furthermore there are 6 choices where the other number is 4 or 5, namely $445, 454, 544, 455, 545, 554$

Therefore there are $48 + 6 = 54$ choices for the numbers.

STEP 3: Hence, by the *mn*-principle, the answer is

$$26 \times 54 = 1404 \text{ choices}$$

(26×48 is completely acceptable)

2. **STEP 1:** Let A be the event "The oracle rolls a 6"

Let B be the event "The oracle says it's a 6"

We want to find $P(A|B)$

STEP 2: By Bayes' Theorem with partition $\{A, A^c\}$ we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$\begin{aligned}
P(B|A) =& \frac{2}{3} \text{ (the oracle tells the truth 2/3 of the time)} \\
P(A) =& \frac{1}{6} \\
P(A^c) =& \frac{5}{6} \\
P(B|A^c) =& \frac{1}{3} \text{ (the oracle lies 1/3 of the time)}
\end{aligned}$$

STEP 3:

$$P(A|B) = \frac{\left(\frac{2}{3}\right)\frac{1}{6}}{\left(\frac{2}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{6}\right)} = \frac{\frac{2}{18}}{\frac{2}{18} + \frac{5}{18}} = \frac{2}{2+5} = \frac{2}{7}$$

Other way:

There was some ambiguity in this problem concerning $P(B|A^c)$, the probability that the oracle says it's a 6 given that you didn't roll a 6, and we also accept the answer below:

Suppose you didn't roll a 6. Then the only way the oracle could say "it's a 6" is if it lied (1/3 chance) and if it chose the number 6 among of the numbers you didn't roll. So for example, if you rolled a 3, then it would choose one of the number 6 among $1, 2, 4, 5, 6$ ($1/5$ chance), hence

$$P(B|A^c) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

$$P(A|B) = \frac{\left(\frac{2}{3}\right)\left(\frac{1}{6}\right)}{\left(\frac{2}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{15}\right)\left(\frac{5}{6}\right)} = \frac{\frac{2}{18}}{\frac{2}{18} + \frac{6}{18}} = \frac{2}{2+6} = \frac{1}{4}$$

3.

STEP 1: By the complement rule, the answer is

(Total) − (Number of groups where the two Bens are in the same group)

STEP 2: Total Number of groups

If the order of the groups mattered, the total number of ways of putting 15 people into 5 groups of 3 would be

$$\binom{15}{3,3,3,3,3} = \frac{15!}{3!3!3!3!3!}$$

But because there are 5! ways of rearranging the groups, the total number of groups is

$$\text{Total} = \frac{1}{5!}\binom{15}{3,3,3,3,3} = \frac{15!}{5!3!3!3!3!3!}$$

STEP 3: Number of groups where the two Bens are in the same group.

Then a typical configuration would look like

(Ben Ben Extra) $(\star\ \star\ \star)$ $(\star\ \star\ \star)$ $(\star\ \star\ \star)$ $(\star\ \star\ \star)$

Assume again that the order of groups mattered.

There are 13 ways of putting in the extra student, and for the other 4 groups it's like putting 12 students in 4 groups of 3, and so the number of configurations would be

$$13 \times \binom{12}{3,3,3,3} = 13\left(\frac{12!}{3!3!3!3!}\right) = \frac{13!}{3!3!3!3!}$$

There are 4! ways or rearranging the other groups, so the total number of configurations where the two Bens are in the same group is

$$\frac{1}{4!}\left(\frac{13!}{3!3!3!3!}\right)$$

STEP 4:

$$\text{Answer: } \frac{15!}{5!3!3!3!3!3!} - \frac{13!}{4!3!3!3!3!}$$

4. **STEP 1:** Let X = number of flavors chosen by both Alice and Bob.

For $i = 1, 2, \cdots, 10$, let X_i be

$$X_i = \begin{cases} 1 & \text{if flavor } i \text{ is chosen by both Alice and Bob} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then } X = X_1 + X_2 + \cdots + X_1 0$$

STEP 2: By properties of expectation, we have

$$E(X) = E(X_1 + \cdots + X_{10}) = E(X_1) + \cdots + E(X_{10})$$

For every i, we have

$$E(X_i) = 0P(X_i = 0) + 1P(X_i = 1) = P(X_i = 1)$$

STEP 3:

Since there is a $\frac{3}{10}$ chance that Alice chooses flavor i and a $\frac{3}{10}$ chance that Bob chooses flavor i, and since they they both choose the flavors independently, we get

$$P(X_i = 1) = \left(\frac{3}{10}\right)\left(\frac{3}{10}\right) = \frac{9}{100}$$

$$\text{Hence } E(X_i) = P(X_i = 1) = \frac{9}{100}$$

STEP 4: Finally, we get

$$\begin{aligned} E(X) =& E(X_1) + \cdots + E(X_{10}) \\ =& \left(\frac{9}{100}\right) + \cdots + \frac{9}{100} \\ =& \left(\frac{9}{100}\right) \times 10 \\ =& \frac{9}{10} \end{aligned}$$

Other way: Some people interpreted "flavors chosen by both Alice and Bob" as the total number of flavors they chose, so in the example it would be Chocolate, Vanilla, Strawberry, Blueberry, Mint. Technically this would be "flavors chosen by Alice **or** Bob," but we accepted this as well. In this case the only change would be

$$\begin{aligned}
P(X_i = 1) =& P(\text{ Alice chose } i \ \cup \ \text{Bob chose } i\ ) \\
=& P(\text{ Alice chose } i\ ) + P(\text{ Bob chose } i\ ) - P(\text{ Alice chose } i \ \cap \ \text{Bob chose } i\ ) \\
=& \frac{3}{10} + \frac{3}{10} - \left(\frac{3}{10}\right)\left(\frac{3}{10}\right) \\
=& \frac{3}{5} - \frac{9}{100} \\
=& \frac{60}{100} - \frac{9}{100} \\
=& \frac{51}{100}
\end{aligned}$$

And we would still have $E(X) = \frac{51}{100} \times 10 = \frac{51}{10} = 5.1$

5. (a) This is like finding 49 heads out of 100 coin tosses, so we use the **binomial distribution** with $n = 100$ trials and success probability $p = \frac{2}{7}$

Let $X \sim$ Binom $\left(100, \frac{2}{7}\right)$, then

$$P(X = 49) = \binom{100}{49} \left(\frac{2}{7}\right)^{49} \left(\frac{5}{7}\right)^{51}$$

(b) Again using the binomial distribution, we want $P(X \geq 3)$ but it's easier to use the complement rule

$$\begin{aligned} P(X \geq 3) =& 1 - P(X \leq 2) \\ =& 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ =& 1 - \binom{100}{0} \left(\frac{2}{7}\right)^{0} \left(\frac{5}{7}\right)^{100} - \binom{100}{1} \left(\frac{2}{7}\right)^{1} \left(\frac{5}{7}\right)^{99} - \binom{100}{2} \left(\frac{2}{7}\right)^{2} \left(\frac{5}{7}\right)^{98} \\ =& 1 - \left(\frac{5}{7}\right)^{100} - 100 \left(\frac{2}{7}\right) \left(\frac{5}{7}\right)^{99} - \binom{100}{2} \left(\frac{2}{7}\right)^{2} \left(\frac{5}{7}\right)^{98} \end{aligned}$$

(c) This is asking about the number of trials until you get your first success, so we use the **geometric distribution** with success probability $p = \frac{2}{7}$

Let $Y \sim$ Geom $\left(\frac{2}{7}\right)$ then

$$P(Y = 40) = (1 - p)^{40-1} p = \left(\frac{5}{7}\right)^{39} \frac{2}{7}$$

(d) Again, using the geometric distribution, we want $E(Y)$ but

$$E(Y) = \frac{1}{p} = \frac{1}{\frac{2}{7}} = \frac{7}{2}$$