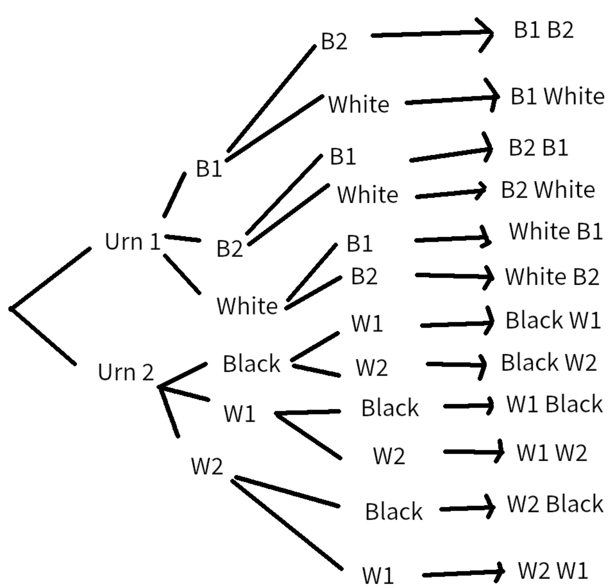
APMA 1650 – MORE PRACTICE QUESTIONS

Note: Those problems taken from past exams from when I taught Discrete Math back in 2016-2017

Problem 1: Suppose there are two urns. Urn 1 contains two black balls (labeled B_1 and B_2) and one white ball. Urn 2 contains one black ball and two white balls (labeled W_1 and W_2). Suppose the following experiment is performed: One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

(a) What is the probability that two black balls are chosen?

The possibilities are as given in the tree diagram below:



So there are 12 total choices. In (a), there are 2 favorable outcomes (B1B2 and B2B1), so the probability is $\frac{2}{12} = \frac{1}{6}$.

(b) What is the probability that two balls of opposite color are chosen?

In (b), there are 8 favorable outcomes (everything except B1B2, B2B1, W1W2, and W2W1), so the probability is $\frac{8}{12} = \frac{2}{3}$.

Problem 2: A telephone number is formed using 7 digits (from 0 to 9). What is the probability that a randomly chosen seven-digit phone number would have at least one repeated digit?

1) Let's find the probability of the event that there are no repeated digits.

$$\underbrace{10}_{\text{10 choices}} \times \underbrace{9}_{\text{9 choices}} \times 8 \times 7 \times 6 \times 5 \times 4$$

The number of favorable outcomes is the number of 7-permutations of 10 objects, or $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$. The total number of outcomes is $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^7$ by the multiplication rule. So the probability is

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10} = \frac{9 \times 7 \times 3}{5 \times 5 \times 5 \times 5 \times 5} = \frac{9 \times 7 \times 3}{5^5} = \frac{3^3 \times 7}{5^5}$$

(It's okay to leave your answer as $\frac{9 \times 7 \times 3}{5 \times 5 \times 5 \times 5 \times 5}$.)

2) By the formula of the probability of the complement, the probability of at least 1 repeated digit is

$$1 - \mathbb{P}(\text{no repeated digit}) = 1 - \frac{3^3 \times 7}{5^5}$$

Problem 3: (there will be no T/F questions on the exam) Label the following statements as **TRUE (T)** or **FALSE (F)**

(a) There are $\frac{9!}{4}$ different ways to arrange the letters in the word TABRIZIAN.

True. There are $\binom{9}{2}$ ways to place the A and $\binom{7}{2}$ ways to place the I. There are 5! ways to place the other letters. Thus, by the multiplication rule, there are

$$\binom{9}{2}\binom{7}{2}5! = \frac{9 \times 8 \times 7 \times 6 \times 5!}{2!2!} = \frac{9!}{2!2!} = \frac{9!}{4}$$

possibilities.

(b) If you toss a fair coin 8 times, the probability of getting ≥ 4 heads is $\frac{1}{2}$.

False.

$$\mathbb{P}(\geq 4 \text{ H}) = 1 - \mathbb{P}(< 4 \text{ H}) = 1 - \mathbb{P}(\leq 3 \text{ H})$$

But

$$\begin{aligned}
\mathbb{P}(\leq 3 \text{ H}) &= \mathbb{P}(\text{no head}) + \mathbb{P}(1 \text{ head}) + \mathbb{P}(2 \text{ heads}) + \mathbb{P}(3 \text{ heads}) \\
&= \binom{8}{0}(\frac{1}{2})^8 + \binom{8}{1}(\frac{1}{2})^8 + \binom{8}{2}(\frac{1}{2})^8 + \binom{8}{3}(\frac{1}{2})^8 \\
&= \frac{1}{2^8}(1 + 8 + 8 \times 7/2 + 8 \times 7 \times 6/6) \\
&= \frac{1}{2^8}(1 + 8 + 28 + 56) \\
&= \frac{93}{256}
\end{aligned}$$

So

$$\mathbb{P}(\geq 4 \text{ H}) = 1 - \frac{93}{256} = \frac{163}{256} > \frac{128}{256} = \frac{1}{2}$$

(c) Suppose you toss a fair coin and you win \$3 if you get H and you win \$1 if you get T, and suppose it costs \$2 to play this game, then the expected value of your game is \$0.

True. The possible outcomes are $(3-2)$ or $(1-2)$, or 1 or -1. The probabilities are $1/2$, so the expectation is

$$\mathbb{E} = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$$

Problem 4: What is the probability that

(a) The top and bottom cards of a randomly shuffled deck of 52 cards are both aces?

We really only care about the top and bottom cards, so the number of total outcomes is $\binom{52}{2}$. The number of favorable outcomes is $\binom{4}{2}$, corresponding to choosing 2 aces from the 4 aces. Therefore, the probability in question is

$$P = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{\frac{4\times 3}{2}}{\frac{52\times 51}{2}} = \frac{4 \times 3}{52 \times 51} = \frac{1}{13 \times 17} = \frac{1}{221}$$

(b) A five-card poker hand contains the ace of hearts?

The total number of outcomes is $\binom{52}{5}$. Here are the favorable outcomes, remembering that the order doesn't matter:

$$\underbrace{\text{(Ace of Hearts)}}_{\text{1 choice}} \underbrace{\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_}_{\text{4 choices out of } 52-1=51}$$

so there are $1 \times \binom{51}{4} = \binom{51}{4}$ such outcomes. Then we get

$$P = \frac{\binom{51}{4}}{\binom{52}{5}}$$

(c) A five-card poker hand contains a full house (= 3 cards of the same **denomination** and 2 cards of the same **denomination**). There are $\binom{52}{5}$ total possible hands.

(a) Denomination of triple: $\binom{13}{1}$

(b) Suit for Triple: $\binom{4}{3}$

(c) Denomination of pair: $\binom{12}{1}$

(d) Suit for pair: $\binom{4}{2}$

So there are

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

favorable outcomes and the final probability is

$$P = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

Problem 5: Suppose that one out of every 1000 Pokémon is shiny, and a Pokéball with the following properties:

- If the Pokémon is shiny, the chances of catching it is 0.1 percent.
- If the Pokémon is not shiny, the chances of catching it is 50 percent.

Prof. Oak tells you that there is a magic potion that doubles the HP of your Pokémon, but the only way to get it is by winning the following game: Suppose you toss a crooked coin with the probability of having H is the probability of a Pokémon being shiny given that you catch it. You win if, out of 10 tosses, you get exactly 3 heads. What is the probability of getting the magic potion?

First let's compute

$$\begin{aligned} p &= \mathbb{P}(\text{shiny}|\text{catch it}) \\ &= \frac{\mathbb{P}(\text{catch it}|\text{shiny})\mathbb{P}(\text{shiny})}{\mathbb{P}(\text{catch it}|\text{shiny})\mathbb{P}(\text{shiny}) + \mathbb{P}(\text{catch it}|\text{not shiny})\mathbb{P}(\text{not shiny})} \\ &= \frac{0.001 \times 0.001}{0.001 \times 0.001 + 0.5 \times 0.999} \\ &\approx 2 \times 10^{-6} \end{aligned}$$

Then

$$\mathbb{P}(\text{exactly 3 H}) = \binom{10}{3} p^3 (1-p)^7$$

because by independence, the order doesn't matter, so we ask outselves in how many ways can we get the combination HHHTTTTTTT, which ends up being $\binom{10}{3}$. The combination HHHTTTTTTT has probability

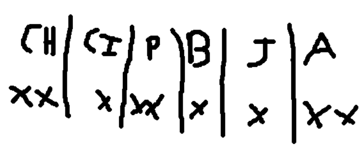
$$\begin{aligned} \mathbb{P}(\text{HHHTTTTTTT}) &= \mathbb{P}(\text{H})\mathbb{P}(\text{H})\mathbb{P}(\text{H})\mathbb{P}(\text{T})\mathbb{P}(\text{T})\mathbb{P}(\text{T})\mathbb{P}(\text{T})\mathbb{P}(\text{T})\mathbb{P}(\text{T})\mathbb{P}(\text{T}) \\ &= ppp(1-p)(1-p)(1-p)(1-p)(1-p)(1-p)(1-p) \\ &= p^3(1-p)^7 \end{aligned}$$

Problem 6: Suppose Dunkin' Donuts offers 6 different kinds of donuts: Chocolate, Cinnamon, Powdered Sugar, Boston Cream, Jelly, and Apple Cider. Today they only have 10 Chocolate donuts left but 40 each of the other kinds of donuts.

(a) How many different selections of 20 donuts are there?

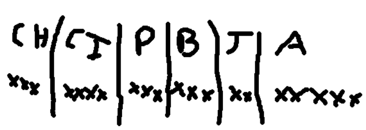
We want to find $|C \leq 10| = \#$ selections with ≤ 10 chocolate donuts. By the complement rule, let's calculate total $- |C_{\geq 11}|$, or the number of selections with at least 11 chocolate donuts. Using the stars and bars method, we have 5 bars and 9 stars.

Here's an example:



Therefore, $|C_{\geq 11}| = \binom{14}{9}$. For the total, we have 5 bars and 20 stars, so there are $\binom{25}{20}$ total options.

Here's an example:



Therefore, $|C_{\leq 10}| = \binom{25}{20} - \binom{14}{9}$.

(b) Suppose in addition to only having 10 Chocolate donuts, it only has 8 Powdered Sugar Donuts. How many different selections of 20 donuts are there?

By the complement rule,

$$\begin{aligned} |C_{\leq 10} \cap P_{\leq 8}| &= \text{Total} - |(C_{\leq 10} \cap P_{\leq 8})^c| \\ &= \binom{25}{20} - |C_{\geq 11} \cup P_{\geq 9}| && \text{by De Morgan} \\ &= \binom{25}{20} - |C_{\geq 11}| - |P_{\geq 9}| + |C_{\geq 11} \cap P_{\geq 9}| && \text{By inclusion-exclusion} \\ &= \binom{25}{20} - \binom{14}{9} - \binom{16}{11} + 1 \end{aligned}$$

where we use the fact $|C_{\geq 11} \cap P_{\geq 9}| = 1$ as there is only one order-invariant selection of 11 chocolate and 9 powdered sugar donuts. $|P_{\geq 9}| = \binom{16}{11}$ following a calculation just like with $|C_{\geq 11}|$; we do stars and bars with 11 stars and 5 bars.

Problem 7: What is the expected value of the number of aces in a 5-card poker hand?

Here's the hard way:

$$\mathbb{P}(\text{exactly } n \text{ aces in a 5-card poker hand}) = \begin{cases} \frac{\binom{4}{n}\binom{48}{5-n}}{\binom{52}{5}} & 0 \leq n \leq 4 \\ 0 & \text{n=5} \end{cases}$$

$$\begin{aligned} E &= \mathbb{E}(\text{number of aces in a 5-card poker hand}) \\ &= \sum_{n=0}^{5} n\mathbb{P}(n \text{ aces in a 5-card poker hand}) \\ &= 1\frac{\binom{4}{1}\binom{48}{5-1}}{\binom{52}{5}} + 2\frac{\binom{4}{2}\binom{48}{5-2}}{\binom{52}{5}} + 3\frac{\binom{4}{3}\binom{48}{5-3}}{\binom{52}{5}} + 4\frac{\binom{4}{4}\binom{48}{5-4}}{\binom{52}{5}} \\ &= \frac{5}{13} \end{aligned}$$

Here's the easy way:

Let Y_i be an indicator variable for the event that the ith card is an ace. Let $Y = \sum_{i=1}^{5} Y_i$. Then we are interested in $\mathbb{E}(Y)$. We can compute

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{i=1}^{5} \mathbb{E}(Y_i) && \text{by linearity of expectation} \\
&= \sum_{i=1}^{5} \mathbb{P}(\text{the } i\text{th card is an ace}) \\
&= \sum_{i=1}^{5} \frac{4}{52} = \frac{20}{52} = \frac{5}{13}
\end{aligned}$$

Problem 8: Suppose you have a bag of 12 crooked coins (6 red and 6 blue). The red ones come up heads $\frac{3}{5}$ of the time and the blue ones come up tails $\frac{2}{3}$ of the time. Suppose you reach into the bag, pull out a coin at random, and toss it. The coin comes up tails. What is the probability that you pulled out a blue coin?

This is similar to Problem 11, through Bayes Theorem and law of total probability:

$$\begin{aligned}
\mathbb{P}(B|T) &= \frac{\mathbb{P}(T|B) * \mathbb{P}(B)}{\mathbb{P}(T|B) * \mathbb{P}(B) + \mathbb{P}(T|R) * \mathbb{P}(R)} \\
&= \frac{(\frac{2}{3}) * (0.5)}{(\frac{2}{3}) * (0.5) + (\frac{2}{5}) * (0.5)} = 0.625
\end{aligned}$$

Problem 9: Oh noes!!! The Dark Lord Bun-ondorf stole Peyam's two fluffy bunnies Oreo and Cookie, and escaped to his evil fortress. You try to enter it, but unfortunately the main door is locked. Next to you, you find a well with the following inscription: "Hey, listen! If you throw in 15 diamonds in this well, then a fairy will appear and will open the door for you. The order you throw in the diamonds doesn't matter, but you need to throw in at least 2 red diamonds and at least

1 blue diamond" Next to you, you also find a bag with 20 Red diamonds, 20 Blue diamonds, 20 Yellow diamonds, 5 Green diamonds, and 5 Purple diamonds. How many different ways can you throw the diamonds in the well?
As you know there must be ≥ 2 red diamonds and ≥ 1 blue diamonds, the question now becomes "How many different ways can you throw in $15 - 2 - 1 = 12$ Diamonds, if you have 18 red diamonds, 19 blue diamonds, 20 yellow diamonds. 5 green diamonds, and 5 purple diamonds."

If there were at least 12 purple and green diamonds then the answer would simply be $\binom{12+5-1}{4}$ since each of the 12 diamonds could be one of 5 colors by using the box and stars approach (See Lecture 4).
Since there cannot be more than 5 green or purple diamonds, our the number of possibilities is instead:

$$\binom{16}{4} - |G_{\geq 6} \cap P_{\geq 6}|$$

Where $G_{\geq 6}, P_{\geq 6}$ are the set of outcomes where there are more than 5 green diamonds and purple diamonds respectively.

From here the use inclusion-exclusion to get that:

$$|G_{\geq 6} \cup P_{\geq 6}| = |G_{\geq 6}| + |P_{\geq 6}| - |G_{\geq 6} \cap P_{\geq 6}|$$

Now $|G_{\geq 6}| = |P_{\geq 6}| = \binom{10}{4}$ by the same logic as before since 6 more of my diamonds are now spoken for, and $|G_{\geq 6} \cap P_{\geq 6}| = 1$ since there is only one way to have 6 green, and 6 purple diamonds with the 12 remaining slots. This gives the final solution:

$$\binom{16}{4} - |G_{\geq 6} \cap P_{\geq 6}| = \binom{16}{4} - (|G_{\geq 6}| + |P_{\geq 6}| - |G_{\geq 6} \cap P_{\geq 6}|)$$

$$= \binom{16}{4} - (\binom{10}{4} + \binom{10}{4} - 1) = \binom{16}{4} - \binom{10}{4} - \binom{10}{4} + 1$$

Possible ways to to toss diamonds while still receiving the fairy's help.
Problem 10: Suppose your poker hand contains at least two aces. What is the probability that it contains all four aces?

This is the same Problem from homework 3.

The best way to approach this problem is using Conditional probability,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Where B is the probability your poker hand has 4 aces, and A is the probability your hand has at least 2 aces in it already.
Luckily $\mathbb{P}(A \cap B)$ is pretty simple since the only time you have at least 2 aces, **and** all 4 aces, is when you have all 4 aces, so $\mathbb{P}(A \cap B) = \mathbb{P}(B)$ so now all we want is:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

Recall $\mathbb{P}(B) = \frac{48}{\binom{52}{5}}$ and the probability of having at least 2 aces in your poker hand is seen my adding the probability of exactly 2,3,and 4 aces, or by taking the compliment which is $1-$(the probability of having 0 or 1) ace. Either way $\mathbb{P}(n \text{ aces}) = \frac{\binom{4}{n} * \binom{48}{5-n}}{\binom{52}{5}}$

Substituting into our above equation:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\frac{48}{\binom{52}{5}}}{\frac{\binom{4}{2} * \binom{48}{3}}{\binom{52}{5}} + \frac{\binom{4}{3} * \binom{48}{2}}{\binom{52}{5}} + \frac{48}{\binom{52}{5}}} = \frac{48}{\binom{4}{2} * \binom{48}{3} + \binom{4}{3} * \binom{48}{2} + 48}$$

Problem 11: Urn 1 contains 10 red balls and 25 green balls, and Urn 2 contains 25 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a crooked coin with probability of 0.4 of landing heads. If the coin lands heads, the Urn 1 is chosen; otherwise Urn 2 is chosen. Then a ball is picked at random from the chosen urn. If the chosen ball is green, what is the probability that it was picked from Urn 1?

Bayes Theorem and law of total probability give:

$$\mathbb{P}(\text{Urn 1}|G) = \frac{\mathbb{P}(G|\text{Urn 1}) * \mathbb{P}(\text{Urn 1})}{\mathbb{P}(G|\text{Urn 1}) * \mathbb{P}(\text{Urn 1}) + \mathbb{P}(G|\text{Urn 2}) * \mathbb{P}(\text{Urn 2})}$$

$$= \frac{(\frac{25}{35}) * (0.4)}{(\frac{25}{35}) * (0.4) + (\frac{15}{40}) * (0.6)}$$

Problem 12: Today, Café Peyam serves Apple, Blueberry, Chocolate, and Yam Pies. How many different selections of 20 pies contain at least 8 Apple Pies and at most 5 Blueberry Pies?

Since there must be 8 apple pies we just care how many combinations of 12 pies there are with at most 5 blueberry pies. Recal problem 9 this total is:

$$\binom{12+4-1}{4-1} - |B_{\geq 6}| = \binom{15}{3} - \binom{6+4-1}{4-1} = \binom{15}{3} - \binom{9}{3}$$

Where $|B_{\geq 6}| isthe$

Problem 13:

(a) Show that

$$\sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{2}\right)^{n-1} = n$$

Hint: First use the binomial theorem to find an expansion of $\left(x+\frac{1}{2}\right)^n$. Then differentiate both sides with respect to x, and finally let $x=\frac{1}{2}$.

By Binomial Theorem with $a=x, b=\frac{1}{2}$ we get:

$$(x+\frac{1}{2})^n = \sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{1}{2}\right)^{n-k}$$

Differentiate both sides

$$n(x+\frac{1}{2})^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} \left(\frac{1}{2}\right)^{n-k}$$

Let $x=\frac{1}{2}$

$$n(\frac{1}{2}+\frac{1}{2})^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right)^{n-k}$$

$$n(1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{1}{2}\right)^{n-1}$$

$$n = \sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{2}\right)^{n-1}$$

(b) Use your answer in (a) to find the expected number of heads when n (fair) coins are tossed.

$$\mathbb{E} = \sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right) \sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right) n = \frac{n}{2}$$

Problem 14: When you are not studying for class, you work part-time as a barista at a local coffee shop. Customers arrive at your coffee shop at an average rate of 10 customers per hour.

(1) Model this problem with an appropriate probability distribution. What is the probability that fewer than 5 customers arrive in a fixed, one-hour period?

Since we have events occurring with a constant average rate in a fixed span of time, and the events occur independently from each other (or at least roughly: two friends could come into your coffee shop together), we will model this with a Poisson distribution with parameter $\lambda = 10$. Let $X \sim$Poisson(10). Then:

$$\begin{aligned}\mathbb{P}(X < 5) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) \\ &= \frac{e^{-10}10^0}{0!} + \frac{e^{-10}10^1}{1!} + \frac{e^{-10}10^2}{2!} + \frac{e^{-10}10^3}{3!} + \frac{e^{-10}10^4}{4!} \\ &\approx 0.029\end{aligned}$$

Don't forget that a Poisson random variable can take the value 0.

(2) What is the probability that 5 or more customers arrive in a fixed, one-hour period?
This is the complementary event from part (a), so we subtract from 1 to get $\mathbb{P}(X \geq 5) = 1 - \mathbb{P}(X < 5) \approx 1 - 0.029 = 0.971$.

Suppose it takes 4 minutes to serve one customer. The total service time is the number of minutes during a fixed, one-hour period which are spent serving customers.

(3) What is the average total service time?

The total service time is just the number of customers multiplied by 4. Let Y be the total service time. Then $Y = 4X$. Using linearity of expectation,

$$\mathbb{E}(Y) = \mathbb{E}(4X) = 4\mathbb{E}(X) = 4(10) = 40$$

where we used the fact that the expected value of a Poisson random variable is its parameter λ, which is 10 in this problem.

(4) What is the variance of the total service time?
Here we use the expression $Var(aX + b) = a^2 Var(X)$.

$$Var(Y) = Var(4X) = 4^2 Var(X) = 16(10) = 160$$

where we used the fact that the variance of a Poisson random variable is also its parameter λ