LECTURE: CONVOLUTION

Today: Laplace transform and multiplication

1. Convolution

Video: Convolution Intuition

Demo: Convolution Demo

WARNING: In general, we have

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

Example 1:

Let $f(t) = 2$ and $g(t) = 3$

$$\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{2 \times 3\} = \mathcal{L}\{6\} = \frac{6}{s}$$

$$\mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = \mathcal{L}\{2\}\,\mathcal{L}\{3\} = \left(\frac{2}{s}\right)\left(\frac{3}{s}\right) = \frac{6}{s^2} \neq \frac{6}{s}$$

However, this property *is* true if we replace multiplication by a new operation called **convolution**

Definition: (Convolution)

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

Here τ is just a dummy variable that gets integrated over. It's really a function of t here

Mnemonic: $(t-\tau)+\tau = \tau + (t-\tau) = t$ (the sum is t)

Note: Think of it as a kind of multiplication of f and g. In fact it *does* become multiplication if you think in terms of Laplace transforms:

Convolution Theorem:

$$\mathcal{L}\{f \star g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

Why? An optional proof is found in the Appendix below. It's based on writing $\mathcal{L}\{f\}\mathcal{L}\{g\}$ as a double integral and changing variables.

We can use this to find Laplace transforms:

Example 2:

$$\text{Find } \mathcal{L}\{h(t)\} \text{ where } h(t) = \int_0^t e^{-2(t-\tau)}\sin(\tau)d\tau$$

Here $h(t) = e^{-2t} \star \sin(t)$ so

$$\mathcal{L}\{h(t)\} = \mathcal{L}\left\{e^{-2t} \star \sin(t)\right\} = \mathcal{L}\left\{e^{-2t}\right\}\mathcal{L}\{\sin(t)\} = \left(\frac{1}{s+2}\right)\left(\frac{1}{s^2+1}\right)$$

Example 3:

Find a function whose Laplace transform is $\frac{1}{(s^2+1)(s^2+4)}$

Note: Write your answer as an integral

Trick: Notice $\frac{1}{s^2+1} = \mathcal{L}\left\{\sin(t)\right\}$ and $\frac{1}{s^2+4} = \mathcal{L}\left\{\frac{1}{2}\sin(2t)\right\}$ and therefore

$$\left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+4}\right) = \mathcal{L}\left\{\sin(t)\right\}\mathcal{L}\left\{\frac{1}{2}\sin(2t)\right\} = \mathcal{L}\left\{\sin(t) \star \left(\frac{1}{2}\sin(2t)\right)\right\}$$

$$\textbf{Answer: } (\sin(t)) \star \left(\frac{1}{2}\sin(2t)\right) = \int_0^t \sin(t-\tau)\left(\frac{1}{2}\sin(2\tau)\right) d\tau$$

2. Convolution and ODE

Convolution allows us to solve ODE where the right hand side isn't part of our Laplace transform table:

Example 4:

$$\begin{cases} y'' - 4y' + 4y = \tan(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

STEP 1: Take Laplace transforms:

$$\begin{aligned} \mathcal{L}\left\{y''\right\} - 4\mathcal{L}\left\{y'\right\} + 4\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{\tan(t)\right\} \\ \left(s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0)\right) - 4\left(s\mathcal{L}\left\{y\right\} - y(0)\right) + 4\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{\tan(t)\right\} \\ \left(s^2 - 4s + 4\right)\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{\tan(t)\right\} \end{aligned}$$

$$\mathcal{L}\left\{y\right\} = \left(\frac{1}{s^2 - 4s + 4}\right)\mathcal{L}\left\{\tan(t)\right\}$$

STEP 2: Write $\frac{1}{s^2-4s+4} = \frac{1}{(s-2)^2}$ as a Laplace transform

This is a shifted version by 2 units of $\frac{1}{s^2} = \mathcal{L}\{t\}$ and so $\frac{1}{(s-2)^2} = \mathcal{L}\left\{e^{2t}t\right\}$

$$\mathcal{L}\{y\} = \mathcal{L}\left\{te^{2t}\right\}\mathcal{L}\{\tan(t)\} = \mathcal{L}\left\{\left(te^{2t}\right)\star(\tan(t))\right\}$$

STEP 3: Solution

$$y = \left(te^{2t}\right)\star(\tan(t)) = \int_0^t (t-\tau)e^{2(t-\tau)}\tan(\tau)d\tau$$

This is unfortunately the best we can do, unless one can evaluate the integral explicitly.

3. Integral Equations

Video: Integral Equations

In fact, we can go a step further and even solve *integral* equations!

Example 5:

Solve the following **integral** equation

$$\phi(t) + \int_0^t (t-\tau)\phi(\tau)d\tau = \frac{8}{3}\sin(3t)$$

The unknown $\phi(t)$ is both under the equation and under the integral.

This is very important in applications: For example the **Lotka-Volterra equations** give us a better model of the bunnies vs foxes situation.

Notice this has the form $\phi + (t\star\phi) = \frac{8}{3}\sin(3t)$

STEP 1: Take Laplace Transforms

$$\begin{aligned}
\mathcal{L}\left\{\phi(t)\right\} + \mathcal{L}\left\{t \star \phi(t)\right\} =& \mathcal{L}\left\{\frac{8}{3}\sin(3t)\right\} \\
\mathcal{L}\left\{\phi(t)\right\} + \mathcal{L}\left\{t\right\}\mathcal{L}\left\{\phi(t)\right\} =& \left(\frac{8}{3}\right)\left(\frac{3}{s^2+9}\right) \\
\mathcal{L}\left\{\phi(t)\right\} + \left(\frac{1}{s^2}\right)\mathcal{L}\left\{\phi(t)\right\} =& \frac{8}{s^2+9} \\
\left(1+\frac{1}{s^2}\right)\mathcal{L}\left\{\phi(t)\right\} =& \frac{8}{s^2+9} \\
\left(\frac{s^2+1}{s^2}\right)\mathcal{L}\left\{\phi(t)\right\} =& \frac{8}{s^2+9} \\
\mathcal{L}\left\{\phi(t)\right\} =& \left(\frac{s^2}{s^2+1}\right)\left(\frac{8}{s^2+9}\right) \\
\mathcal{L}\left\{\phi(t)\right\} =& \frac{8s^2}{\left(s^2+1\right)\left(s^2+9\right)}
\end{aligned}$$

STEP 2: Partial Fractions

$$\frac{8s^2}{\left(s^2+1\right)\left(s^2+9\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}$$

And eventually you find $A = 0, B = -1, C = 0, D = 9$

STEP 3:

$$\mathcal{L}\left\{\phi(t)\right\} = \frac{-1}{s^2+1} + \frac{9}{s^2+9} = \mathcal{L}\left\{-\sin(t) + 3\sin(3t)\right\}$$

$$\phi(t) = -\sin(t) + 3\sin(3t)$$

Video: Integro-Differential Equation

Example 6: (more practice)

Solve the following **integral** equation with $\phi(0) = 1$

$$\phi'(t) - \frac{1}{2}\int_0^t (t-\tau)^2\,\phi(\tau)d\tau = -t$$

Notice this is of the form $\phi' - \frac{1}{2}(t^2 \star \phi) = -t$

Take Laplace transforms

$$\begin{aligned}
\mathcal{L}\left\{\phi'\right\} - \frac{1}{2}\mathcal{L}\left\{t^2 \star \phi\right\} =& \mathcal{L}\left\{-t\right\} \\
s\mathcal{L}\left\{\phi\right\} - \phi(0) - \frac{1}{2}\mathcal{L}\left\{t^2\right\}\mathcal{L}\left\{\phi\right\} =& -\frac{1}{s^2} \\
s\mathcal{L}\left\{\phi\right\} - 1 - \frac{1}{2}\left(\frac{2}{s^3}\right)\mathcal{L}\left\{\phi\right\} =& -\frac{1}{s^2} \\
\left(s - \frac{1}{s^3}\right)\mathcal{L}\left\{\phi\right\} =& 1 - \frac{1}{s^2} \\
\left(\frac{s^4-1}{s^3}\right)\mathcal{L}\left\{\phi\right\} =& \frac{s^2-1}{s^2} \\
\mathcal{L}\left\{\phi\right\} =& \left(\frac{s^3}{s^4-1}\right)\left(\frac{s^2-1}{s^2}\right) \\
\mathcal{L}\left\{\phi\right\} =& \frac{s\left(s^2-1\right)}{s^4-1}
\end{aligned}$$

To factor out the denominator, notice $s^4 - 1 = \left(s^2\right)^2 - 1^2 = \left(s^2-1\right)\left(s^2+1\right)$

$$\mathcal{L}\left\{\phi\right\} = \frac{s\left(s^2-1\right)}{\left(s^2-1\right)\left(s^2+1\right)} = \frac{s}{s^2+1} = \mathcal{L}\left\{\cos(t)\right\}$$

$$\phi(t) = \cos(t)$$

4. The Laplast One

Video: Laplace integral gone bananas

Finally, here is a cool application of convolution with integrals

Example 7:

$$I = \int_0^1 x^3 (1-x)^7 \, dx$$

STEP 1: Here's a really cool trick: Let $f(t) = t^3$ and $g(t) = t^7$

$$\text{Calculate } f \star g = \int_0^t \tau^3 (t-\tau)^7 d\tau$$

(Notice $\tau + (t - \tau) = t$)

STEP 2: u-substitution

We want to change the endpoint t to 1, so let $u = \frac{\tau}{t}$ then $u(0) = \frac{0}{t} = 0$ and $u(t) = \frac{t}{t} = 1$ and $\tau = tu$ and $d\tau = tdu$ and so

$$\begin{aligned} f \star g &= \int_0^1 (tu)^3 (t - tu)^7 \, tdu \\ &= \int_0^1 (tu)^3 (t(1-u))^7 \, tdu \\ &= t^{3+7+1} \underbrace{\int_0^1 u^3 (1-u)^7 du}_{I} \\ &= t^{11} (I) \end{aligned}$$

STEP 3: Take Laplace transforms on both sides:

$$\begin{aligned}
\mathcal{L}\left\{f \star g\right\} =& \mathcal{L}\left\{t^{11}\left(I\right)\right\} \\
\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\} =& I\mathcal{L}\left\{t^{11}\right\} \\
\mathcal{L}\left\{t^{3}\right\}\mathcal{L}\left\{t^{7}\right\} =& I\mathcal{L}\left\{t^{11}\right\} \\
\left(\frac{3!}{\cancel{s^{4}}}\right)\left(\frac{7!}{\cancel{s^{8}}}\right) =& I\left(\frac{11!}{\cancel{s^{12}}}\right) \\
3!\,7! =& I\left(11!\right) \\
I =& \frac{3!\,7!}{11!}
\end{aligned}$$

STEP 4: Conclusion

$$I = \int_0^1 x^3(1-x)^7 dx = \frac{3!7!}{11!} = \frac{1}{1320}$$

In general:

$$\int_0^1 x^m(1-x)^n dx = \frac{m!\,n!}{(m+n+1)!}$$

5. Appendix: Convolution Theorem Proof

Convolution Theorem:

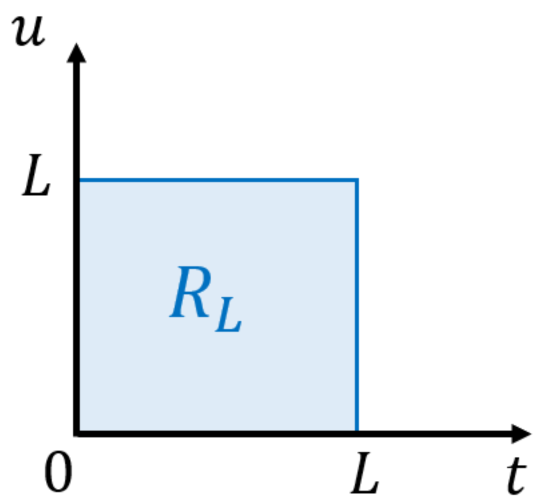
$$\mathcal{L}\left\{f \star g\right\} = \mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}$$

Why?

STEP 1:

$$\begin{aligned}
\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\} &= \left(\int_0^\infty f(t)e^{-st}dt\right)\left(\int_0^\infty g(u)e^{-su}du\right) \\
&= \lim_{L\to\infty}\left(\int_0^L f(t)e^{-st}dt\right)\left(\int_0^L g(u)e^{-su}du\right) \\
&= \lim_{L\to\infty}\int_0^L\left(\int_0^L f(t)e^{-st}dt\right)g(u)e^{-su}du \\
&= \lim_{L\to\infty}\int_0^L\int_0^L f(t)e^{-st}g(u)e^{-su}dtdu \\
&= \lim_{L\to\infty}\int\int_{R_L} e^{-s(t+u)}f(t)g(u)dudt
\end{aligned}$$

Where R_L is the square $0 \leq t \leq L$ and $0 \leq u \leq L$

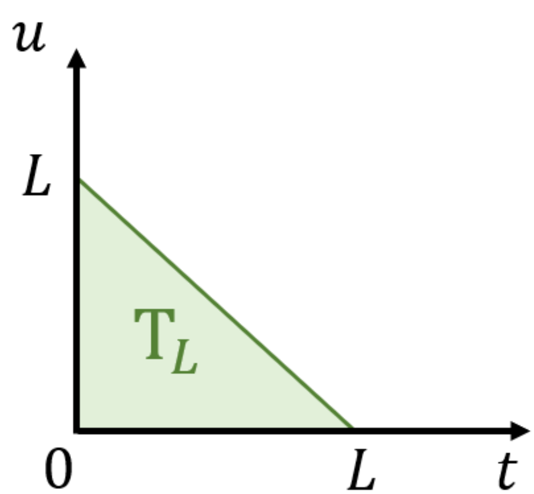


STEP 2: **Trick:** Change the domain of integration

Note that *in the limit* as $L \to \infty$, the integral is the same as

$$\lim_{L\to\infty} \int\int_{R_L} e^{-s(t+u)} f(t)g(u)dudt = \lim_{L\to\infty} \int\int_{T_L} e^{-s(t+u)} f(t)g(u)dudt$$

Where T_L is the triangular region $0 \leq t \leq L$ and $0 \leq t+u \leq L$



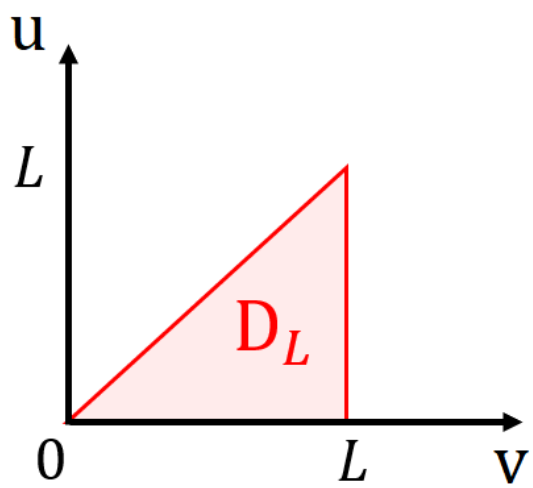
This is because, in the limit, both regions equal to the first quadrant

STEP 3: Change of Variables

$$\begin{cases} u = & u \\ v = & u + t \end{cases} \Rightarrow \begin{cases} u = & u \\ t = & v - u \end{cases}$$

This change of variables turns T_L into D_L

Where D_L is the triangular region $0 \leq v - u \leq L$ and $0 \leq u \leq L$



Jacobian:

$$dudt = \left| \frac{dudt}{dudv} \right| dudv = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} dudv = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} dudv = |1| \, dudv = dudv$$

$$\begin{aligned}
\int\int_{T_L} e^{-s(t+u)} f(t)g(u)dudt &= \int\int_{D_L} e^{-sv} f(v-u)g(u)dudv \\
&= \int_0^L \int_0^v e^{-sv} f(v-u)g(u)dudv \\
&= \int_0^L e^{-sv} \left(\int_0^v f(v-u)g(u)du\right) dv \\
&= \int_0^L e^{-sv} \left(f \star g\right)(v)dv
\end{aligned}$$

STEP 4: Conclusion

Therefore, in the limit as $L \to \infty$, we get

$$\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\} = \lim_{L\to\infty} \int_0^L e^{-sv} \left(f \star g\right)(v)dv = \int_0^\infty e^{-sv} \left(f \star g\right)(v)dv = \mathcal{L}\left\{f \star g\right\}$$