LECTURE: LINEAR ALGEBRA REVIEW

Today: Review of eigenvalues and eigenvectors, because they will be our main tool to solve systems of ODE (our next topic)

1. Eigenvalues and Eigenvectors

Video: Eigenvalues and Eigenvectors

Example 1: (Motivation)

Consider $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (6)(1) \\ (5)(1) + (2)(1) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \mathbf{v}$$

$A\mathbf{v}$ isn't just random, but in fact a multiple of $\mathbf{v}$. In this case, we call $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector of A and $\lambda = 7$ (the multiple) an eigenvalue:

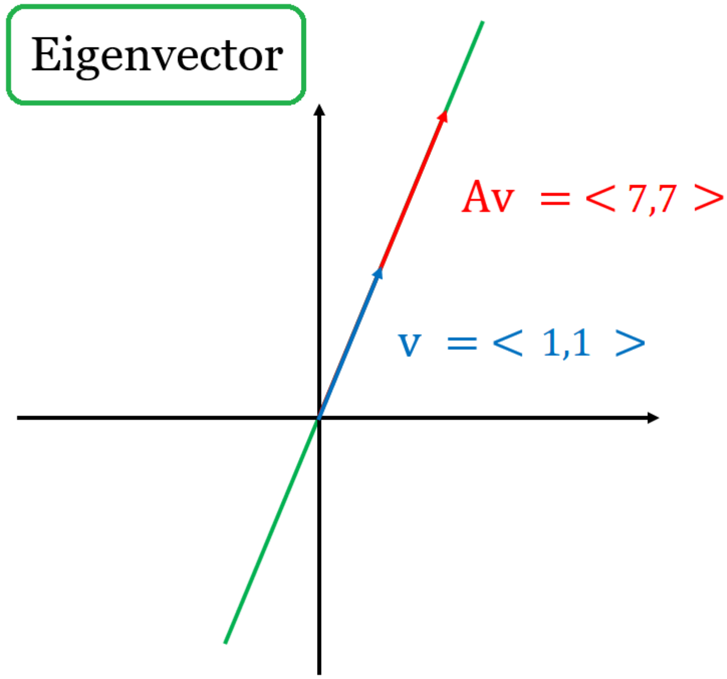
Definition:

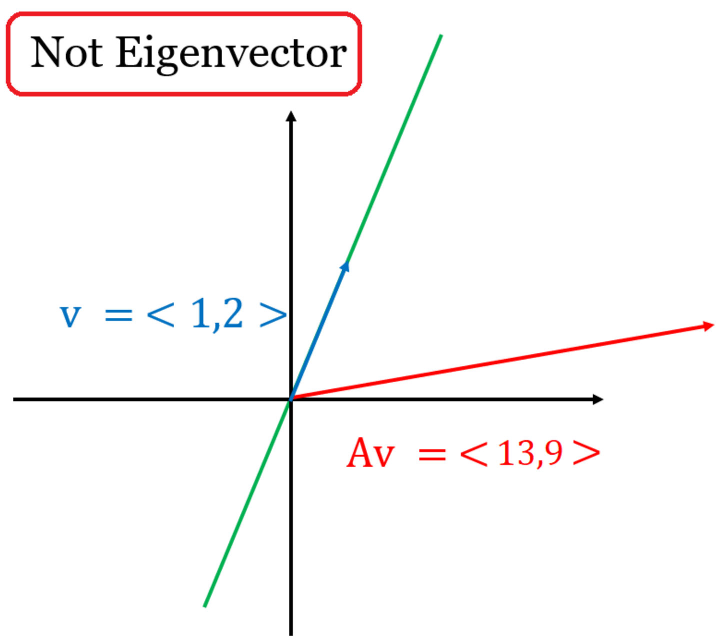
If $A\mathbf{v} = \lambda \mathbf{v}$ then:

λ is called an **eigenvalue** of A

$\mathbf{v}$ is an **eigenvector** of A corresponding to λ

Interpr: If $\mathbf{v}$ is an eigenvector, then $\mathbf{v}$ and $A\mathbf{v}$ lie on the same line!


Eigenvector
Av = < 7,7 >
v = < 1,1 >


Not Eigenvector
v = < 1,2 >
Av = < 13,9 >

Application 1: Resonance

Suppose you're walking on a bridge. Then you have your own frequency λ whereas the bridge has its natural frequency λ' Usually they are different so it's not a big issue. But if it happens that $\lambda = \lambda'$ then there is resonance and the bridge collapses like in the following video: Resonance Effect

Application 2: Music

Whenever you play a musical instrument, the eigenvalues λ are called the **frequencies** of your system and are the sounds you hear. The smallest eigenvalue is the **principal harmonic** (the main sound) and the other ones are the **overtones**

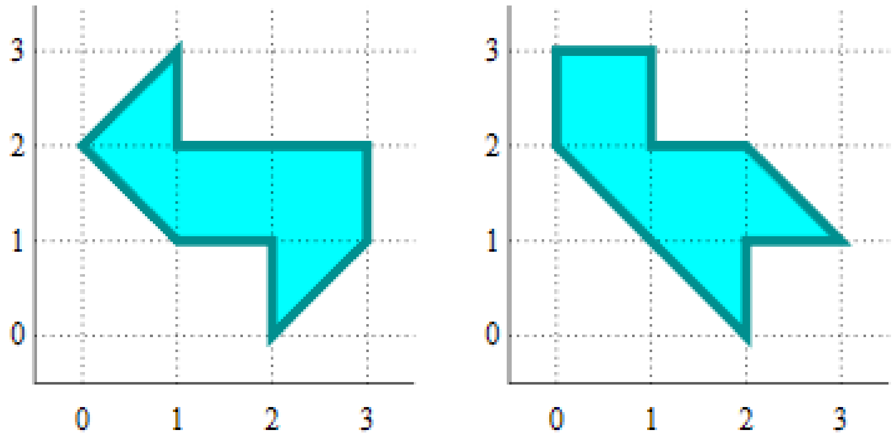
An famous question posed by Marc Kac is "Can you hear the shape of the drum?"

In other words, if you only know the eigenvalues of the instrument, then can you tell me what the shape of the instrument looks like?

So if you're in a room and you hear someone in another room playing an instrument, can you tell me exactly what that instrument is?

YES in 2 dimensions if the instrument is smooth (and convex)

NO if the instrument has corners. In fact, here are two different shapes with the same sound:



NO in higher dimensions: There is a 16−dimensional counterexample (with donuts)

Joke: Q: What did the Linear Algebra motivational speaker say to their audience? A: "If eigendoit, then so can you!"

2. Finding Eigenvalues

Question: How to find eigenvalues?

Example 2:

Find the eigenvalues of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

Motivation:

$$\begin{aligned} &A\mathbf{v} = \lambda \mathbf{v} \\ \Rightarrow &A\mathbf{v} - \lambda \mathbf{v} = \mathbf{0} \\ \Rightarrow &A\mathbf{v} - \lambda I \mathbf{v} = \mathbf{0} \\ \Rightarrow &(A - \lambda I)\mathbf{v} = \mathbf{0} \\ \Rightarrow &\det(A - \lambda I) = 0 \end{aligned}$$

Here $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix and $|A - \lambda I|$ is the determinant.

Mnemonic: $A - \lambda I$ looks like Ali (as in Muhammad Ali)

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det \begin{bmatrix} 1-\lambda & 6-0 \\ 5-0 & 2-\lambda \end{bmatrix} \\
&= \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)(2-\lambda) - 5(6) \\
&= 2 - \lambda - 2\lambda + \lambda^2 - 30 \\
&= \lambda^2 - 3\lambda - 28 \\
&= (\lambda - 7)(\lambda + 4) \\
&= 0
\end{aligned}$$

$$\lambda = 7 \text{ or } \lambda = -4$$

Example 3: (more practice)

Find the eigenvalues of $A = \begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix}$

$$
\begin{aligned}
\det(A-\lambda I) = & \begin{vmatrix} -\lambda & 6 \\ -1 & 5-\lambda \end{vmatrix} \qquad \text{(Subtract } \lambda \text{ from the diagonals)} \\
= & (-\lambda)(5-\lambda) - 6(-1) \\
= & -5\lambda + \lambda^2 + 6 \\
= & \lambda^2 - 5\lambda + 6 \\
= & (\lambda-2)(\lambda-3) \\
= & 0
\end{aligned}
$$

Which gives $\boxed{\lambda = 2 \text{ or } \lambda = 3}$

3. Finding Eigenvectors

Question: How do we find eigenvectors?

Example 4:

Find the eigenvectors of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

STEP 1: Find the eigenvalues: $\lambda = 7$ and $\lambda = -4$

STEP 2: $\lambda = 7$

Motivation: $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$

Strategy: For every λ you found, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Note: This is sometimes called the nullspace, $\text{Nul}\,(A - \lambda I)$

$$\begin{aligned}
\text{Nul } (A - 7I) &= \left[\begin{array}{cc|c} 1-7 & 6 & 0 \\ 5 & 2-7 & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right] \\
&\xrightarrow{(\div -6)R_1 \ (\div 5)R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

This says $x - y = 0$ and so $x = y$ and

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 7$

Important: You should **never** find $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$!! If you do, you either found the wrong eigenvalue, or you made a mistake in your row reduction!

STEP 3: $\lambda = -4$

$$\begin{aligned}
\text{Nul } (A - (-4)I) &= \left[\begin{array}{cc|c} 1-(-4) & 6 & 0 \\ 5 & 2-(-4) & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} 5 & 6 & 0 \\ 5 & 6 & 0 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|c} 5 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$5x + 6y = 0$$

Trick: Find *one* value of x and y that works, for example $x = -6$ and $y = 5$. This is because all the eigenvectors lie on one line, they are multiples of each other.

$$\mathbf{v} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

So an eigenvector for $\lambda = -4$ is $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$

Note: In general, for $ax + by = 0$ you can guess $x = -b$ and $y = a$

Note: It is ok to multiply an eigenvector by any (nonzero) number so if you found $\begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}$ it's ok to multiply this by 5 to get $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$

4. Diagonalization

Usually you see the above question worded differently:

Example 5:

Find D diagonal and P such that $A = PDP^{-1}$ where

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$\lambda = 7 \text{ or } \lambda = -4 \Rightarrow D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \qquad \text{Diagonal}$$

Note: Ok to write $D = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}$ as long as you put in the eigenvectors in the correct order.

For $\lambda = 7$ we found $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and for $\lambda = -4$ we found $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$ and so

$$P = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}$$

Here the first column of P *has* to be an eigenvector for $\lambda = 7$ and the second column has to be an eigenvector for $\lambda = -4$ since we chose D to be in that order.

Why? This is because

$$\begin{aligned} AP &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix} = \left[\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -6 \\ 5 \end{bmatrix} \right] \\ &= \left[7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -4 \begin{bmatrix} -6 \\ 5 \end{bmatrix} \right] = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} = PD \end{aligned}$$

$$AP = PD \Rightarrow APP^{-1} = PDP^{-1} \Rightarrow A = PDP^{-1}$$

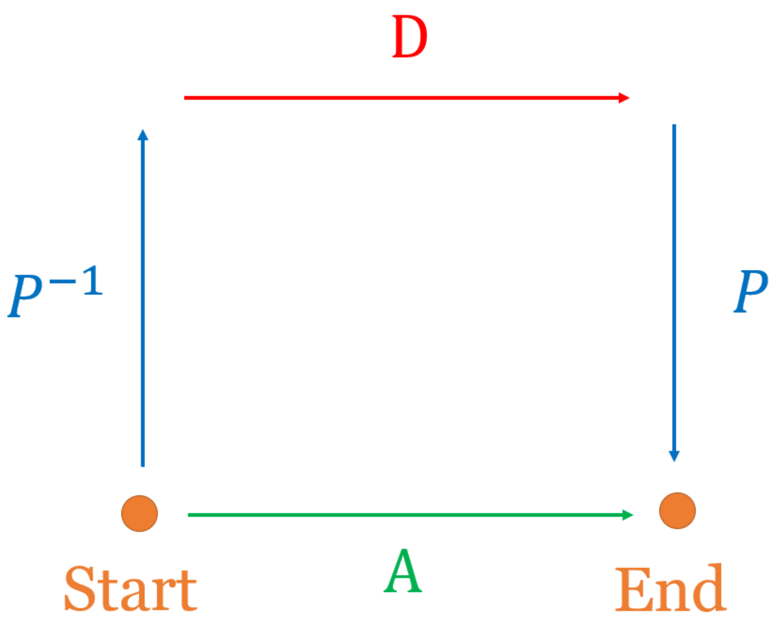
Interpretation: $A = PDP^{-1}$ means that A is "similar to" or "like" D

5. Analogy

> **Video:** Diagonalization and Legend of Zelda

Question: Why does $A = PDP^{-1}$ imply that A is "like" D?

Analogy: Suppose you want to fly from one city to another one. Then you have two options:



Option 1: Take a direct flight A

Option 2: Take a layover flight P^{-1} then another flight D and then another layover flight P. This flight is called PDP^{-1} (you read it from right to left, just like functions)

$A = PDP^{-1}$ means that those two options bring you to the same destination. And in fact if you ignore the layovers P^{-1} and P, then A is "like" D, they're similar flights.

Another Analogy: Legend of Zelda-Twilight Princess

In this game in order to teleport yourself from Hyrule to Kokoriko (A), you first transform yourself into a wolf (P^{-1}) then teleport yourself as a wolf (D) and then transform yourself back (P), see this video.

This whole process is called PDP^{-1}. And if we ignore the Wolf-Link transformation P, then teleporting yourself as Link (A) is the same as teleporting yourself as a wolf (D). This is why A is similar to D.