LECTURE: SYSTEMS OF ODE (I)

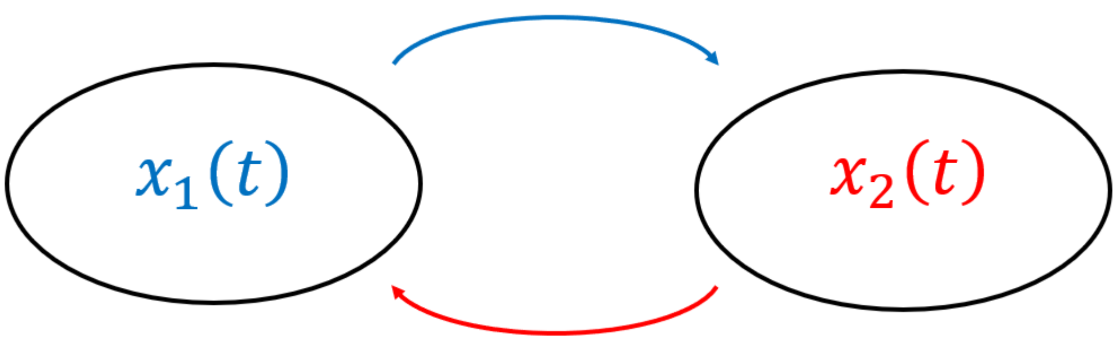
1. Introduction

Welcome to the magical world of systems of ODE! They are equations where functions not only depend on themselves, but on each other

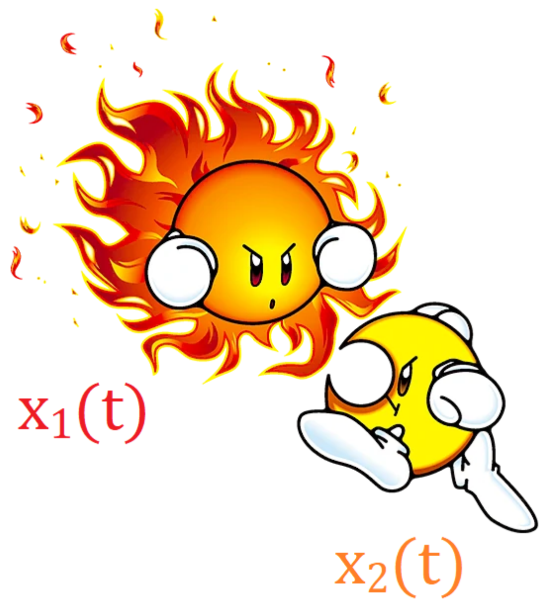
Example 1: (Model Problem)

$$\begin{cases} x_1'(t) = 7x_1(t) - 3x_2(t) \\ x_2'(t) = 10x_1(t) - 4x_2(t) \end{cases}$$

$x_1 = x_1(t)$ and $x_2 = x_2(t)$ are two functions of time. Think for instance two particles colliding with each other, like the sun and the moon



So in some sense, those functions are coupled together, and our job is to disentangle them

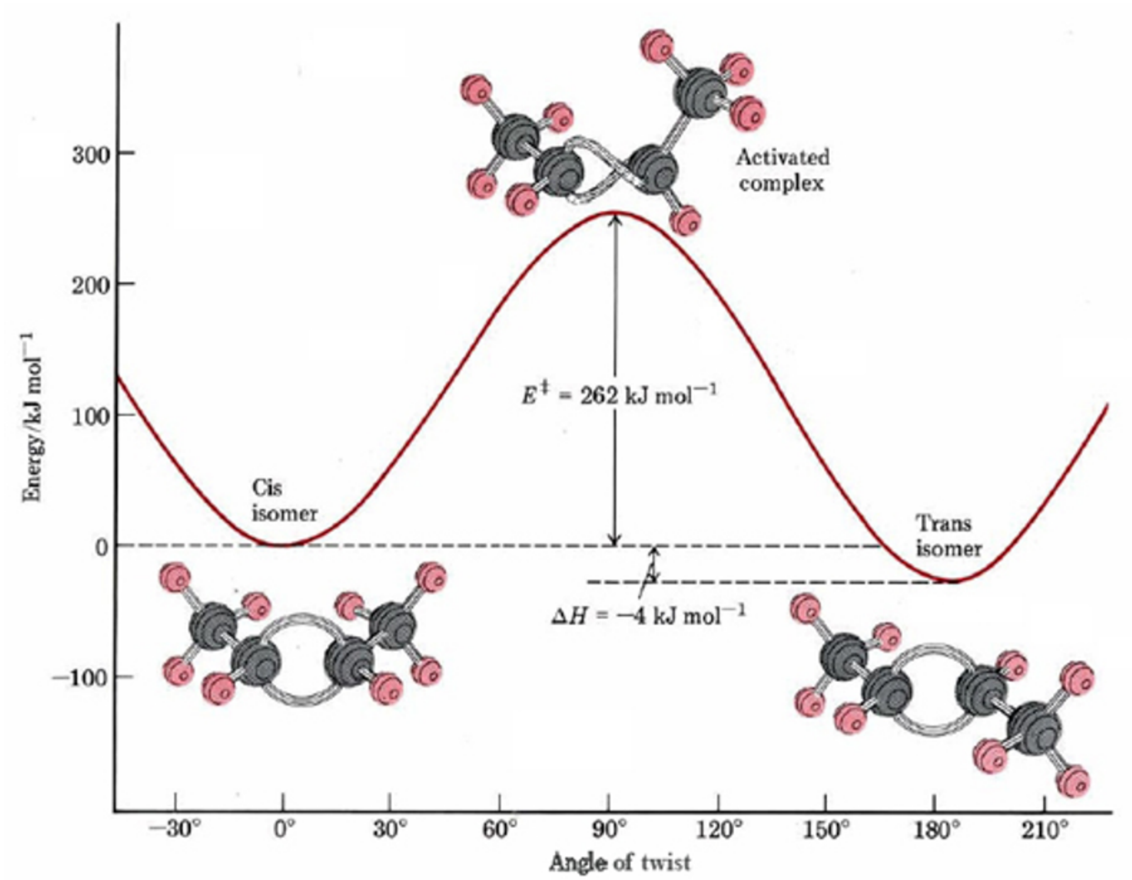


2. Application 1: Chemical Reactions

Systems of ODE arise in many different fields such as astrophysics, ecology, or chemistry. It should therefore come to no surprise that even my PhD thesis involved a of system of differential equations.

In fact, let me briefly tell you where systems appear in my thesis. It has to do with chemistry:

Consider a chemical reaction $A \rightleftharpoons B$ where a molecule A turns into B and vice-versa. Think A twisting to become B:



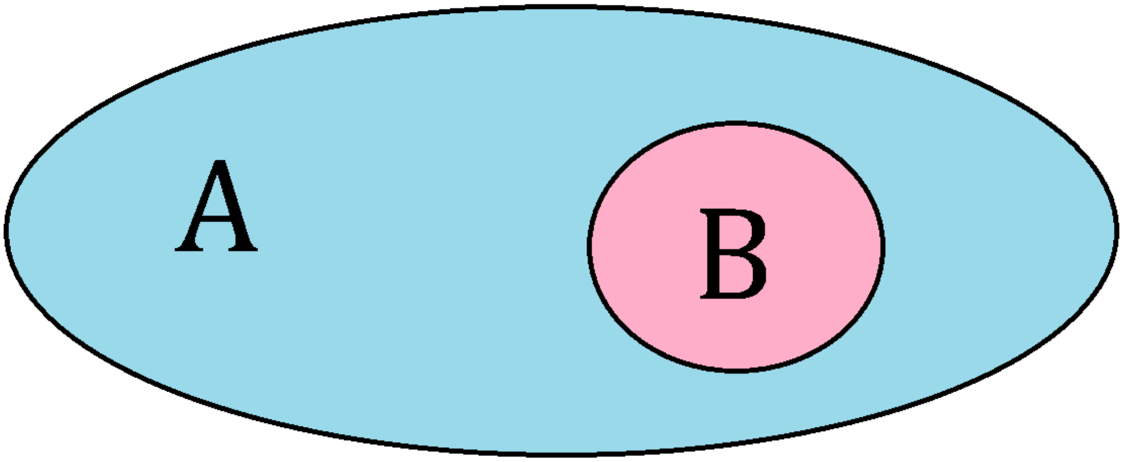
Then if:

$$\begin{cases} x_1(t) = \text{concentration of } A \text{ at } t \\ x_2(t) = \text{concentration of } B \text{ at } t \end{cases}$$

Then the chemical reaction can be modeled by a system of **reaction diffusion equations**:

$$\begin{cases} x_1'(t) = x_2(t) - x_1(t) \\ x_2'(t) = x_1(t) - x_2(t) \end{cases}$$

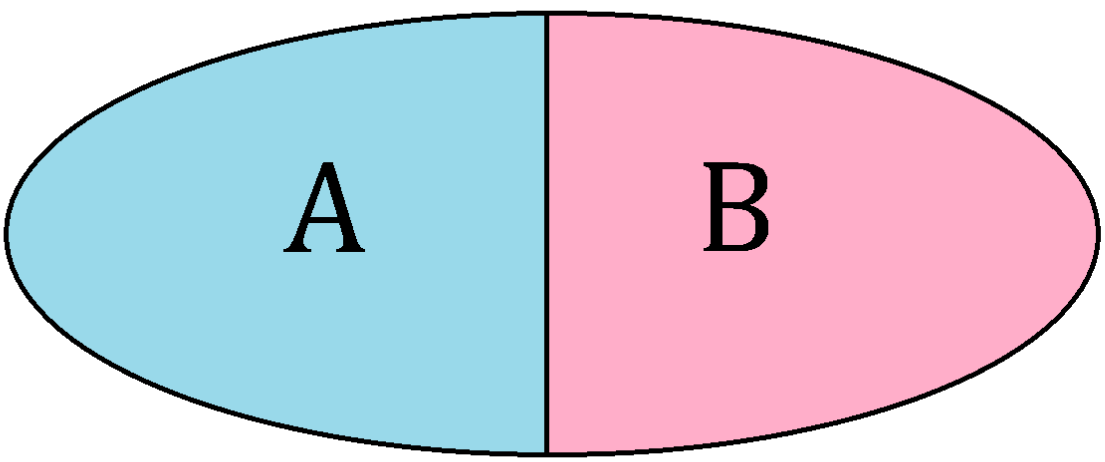
Intuitively: This makes sense because if you have a compound with lots of A and not a lot of B, as in the following picture:



Then $x_1 > x_2$ and therefore

$$x_1'(t) = \underbrace{x_2(t) - x_1(t)}_{<0} < 0 \Rightarrow x_1(t) \text{ decreases } \Rightarrow A \text{ decreases}$$

And similarly B increases until we get equilibrium:



This motivates systems of differential equations quite nicely because in the example above, x_1 and x_2 are coupled together.

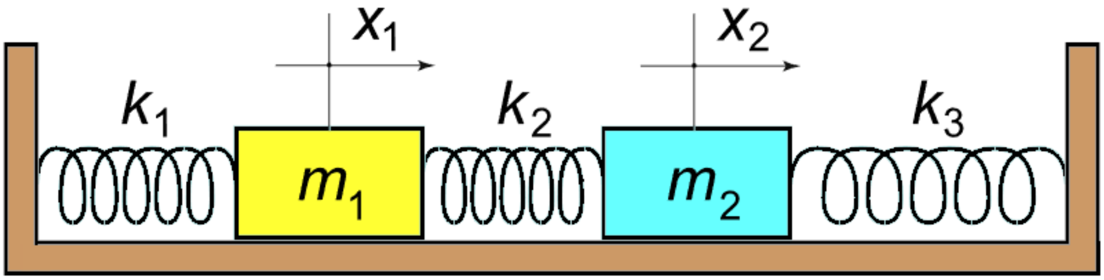
Note: If you're interested in learning more about my thesis, check out the following video:

Video: The PDE that got me the PhD

3. Application 2: Double Mass Spring System

Demo: Double Mass Spring

Consider two masses on three springs, as follows:[1]



In this case, the displacements x_1 and x_2 can be modeled by

$$\begin{cases} m_1\, x_1''(t) = -(k_1 + k_2)x_1 + k_2 x_2 + F_1(t) \\ m_2\, x_2''(t) = k_2 x_1 - (k_2 + k_3)x_2 + F_2(t) \end{cases}$$

[1] Picture taken from this website

Here $F_1(t)$ and $F_2(t)$ are external forces that are applied to the masses, and k_1, k_2, k_3 are called spring constants. It's an example of an **inhomogeneous system**

Note: Just like we can study a single pendulum in physics, we can also consider a double pendulum, like in the demo below. In this case the motion is chaotic; it's one of the easiest examples of a chaotic ODE.

Demo: Double Pendulum

4. Application 3: Pokemon Battle!

Video: Pokémon Battle

Even videogame fights can be modeled by systems of ODE, since two or more characters are interacting with each other

$$\begin{cases} x_1(t) = \text{HP of Charmander at } t \\ x_2(t) = \text{HP of Pikachu at } t \end{cases}$$

Then one possible battle can be given by:

$$\begin{cases} x_1'(t) = 16x_1(t) - 35x_2(t) \\ x_2'(t) = 6x_1(t) - 13x_2(t) \end{cases}$$

Here is what the numbers mean concretely:

- $16x_1(t)$ means that Charmander uses healing
- $-35x_2(t)$ means Charmander gets damage from Pikachu
- $6x_1(t)$ means Pikachu steals HP from Charmander
- $-13x_2(t)$ means Pikachu electrocutes itself

We'll be able to show that Charmander loses fairly quickly.

Note: For a game like Super Smash Bros Ultimate with 8 players, you could represent that by an 8×8 system

5. Matrix Form

Let's now go back to our main problem

Example 2: (Model Problem)

$$\begin{cases} x_1'(t) = 7x_1(t) - 3x_2(t) \\ x_2'(t) = 10x_1(t) - 4x_2(t) \end{cases}$$

In order to solve this, the first step is to write this in "matrix" form.

For now it just means put the coefficients in a table

$$\underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{\mathbf{x}'(t)} = \underbrace{\begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{x}(t)}$$

At first sight this seems to be a cosmetic fix, but we'll see later why this is so useful.

Moral: From now on, systems will be written as $\mathbf{x}'(t) = A\mathbf{x}(t)$

Note: Compare this with $y' = ay$ from the beginning of the course

Example 3:

$$\begin{cases} x_1'(t) = x_1(t) - 2x_2(t) & +t^2 \\ x_2'(t) = 4x_1(t) & +2t \end{cases}$$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

Here we have an **inhomogeneous system** of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$

In fact the key to solving systems of ODE is linear algebra, and we will see how to do this next time!

6. Higher Order Equations

The cool thing is that we can actually turn any higher order ODE into a system, as follows:

Example 4:

Write $y'' + 4y = 0$ as a system of differential equations

Trick: Let $x_1(t) = y$ and $x_2(t) = y'$ then:

$$\begin{cases} x_1'(t) = y' = x_2(t) \\ x_2'(t) = y'' = -4y = -4x_1(t) \end{cases}$$

$$\begin{cases} x_1'(t) = 0x_1(t) + 1x_2(t) \\ x_2'(t) = -4x_1(t) + 0x_2(t) \end{cases}$$

$\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

Example 5:

Same but $y''' - 3y'' + 3y' - y = t^2$

$x_1(t) = y$ and $x_2(t) = y'$ and $x_3(t) = y''$

$$\begin{cases} x_1'(t) = y' = x_2(t) \\ x_2'(t) = y'' = x_3(t) \\ x_3'(t) = y''' = 3y'' - 3y' + y + t^2 = 3x_3(t) - 3x_2(t) + x_1(t) + t^2 \end{cases}$$

$$\begin{cases} x_1'(t) = 0x_1(t) + 1x_2(t) + 0x_3(t) + 0 \\ x_2'(t) = 0x_1(t) + 0x_2(t) + 1x_3(t) + 0 \\ x_3'(t) = 1x_1(t) - 3x_2(t) + 3x_3(t) + t^2 \end{cases}$$

(The order matters here)

$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \qquad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ t^2 \end{bmatrix}$$

This is an example of a 3×3 system