APMA 0350 – EXAM 2 – SOLUTIONS

1. **Auxiliary Equation:** $r^2 = \lambda$

Case 1: $\lambda > 0$

Then $r^2 = \lambda = \omega^2$ and so $r = \pm\omega$

$$y = Ae^{\omega t} + Be^{-\omega t}$$
$$y(0) = A + B = 0 \Rightarrow B = -A$$
$$y = Ae^{\omega t} - Ae^{-\omega t}$$
$$y' = A\omega e^{\omega t} - A(-\omega)e^{-\omega t}$$

$$\begin{aligned} y'(2\pi) &= 0 \\ A\omega e^{2\pi\omega} + A\omega e^{-2\pi\omega} &= 0 \\ \cancel{A\omega} e^{2\pi\omega} &= -\cancel{A\omega} e^{-2\pi\omega} \\ \underbrace{e^{2\pi\omega}}_{>0} &= \underbrace{-e^{-2\pi\omega}}_{<0} \end{aligned}$$

Which is a contradiction $\Rightarrow\Leftarrow$

Case 2: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (repeated twice)

$$y = A + Bt$$

$$y(0) = 0 \Rightarrow A = 0$$
$$y = Bt$$
$$y'(t) = B$$
$$y'(2\pi) = 0 \Rightarrow B = 0$$

But then in this case $y = 0 \Rightarrow\Leftarrow$

Case 3: $\lambda < 0$

In this case $\lambda = -\omega^2$ where $\omega > 0$

Aux: $r^2 = \lambda = -\omega^2 \Rightarrow r = \pm \omega i$

$$y = A\cos(\omega t) + B\sin(\omega t)$$

$$y(0) = A = 0$$

$$y = B\sin(\omega t)$$

$$y' = \omega B\cos(\omega t)$$

$$\begin{aligned} y'(2\pi) &= 0 \\ \not{\omega}\not{B}\cos(2\pi\omega) &= 0 \\ \cos(2\pi\omega) &= 0 \\ 2\pi\omega &= \frac{\pi}{2} + \pi m \\ \omega &= \frac{1}{4} + \frac{m}{2} \qquad m = 0, 1, 2, \cdots \end{aligned}$$

Answer:

Eigenvalues:

$$\lambda = -\omega^2 = -\left(\frac{1}{4} + \frac{m}{2}\right)^2 \qquad m = 0, 1, 2, \cdots$$

Eigenfunctions:

$$y = \sin(\omega t) = \sin\left(\left(\frac{1}{4} + \frac{m}{2}\right)t\right) \qquad m = 0, 1, 2 \cdots$$

2. **STEP 1: Homogeneous Solution**

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$
$$y_0 = A\cos(2t) + B\sin(2t)$$

STEP 2: Particular Solution

$\sin(2t)$ corresponds to $r = \pm 2i$ which coincides with one of the roots of the auxiliary equation.

Hence there is resonance and we must guess

$$y_p = At\cos(2t) + Bt\sin(2t)$$
$$(y_p)' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$$
$$\begin{aligned}(y_p)'' &= -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t)\\ &= -4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t)\end{aligned}$$

$$(y_p)'' + 4(y_p) = 4\sin(2t)$$
$$-4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t) + 4(At\cos(2t) + Bt\sin(2t)) = 4\sin(2t)$$
$$-4A\sin(2t) - \cancel{4At\cos(2t)} + 4B\cos(2t) - \cancel{4Bt\sin(2t)} + \cancel{4At\cos(2t)} + \cancel{4Bt\sin(2t)} = 4\sin(2t)$$
$$-4A\sin(2t) + 4B\cos(2t) = \sin(2t) + 0\cos(2t)$$

$$\begin{cases} -4A = 4 \\ 4B = 0 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 0 \end{cases}$$

$$y_p = -t\cos(2t)$$

STEP 3: General Solution

$$y = y_0 + y_p = A\cos(2t) + B\sin(2t) - t\cos(2t)$$

STEP 4: Initial Condition

$$\begin{aligned} y(0) &= 0 \\ A\cos(0) + B\sin(0) - 0\cos(0) &= 0 \\ A &= 0 \end{aligned}$$

$$y = B\sin(2t) - t\cos(2t)$$
$$y' = 2B\cos(2t) - \cos(2t) + 2t\sin(2t)$$
$$\begin{aligned} y'(0) &= 7 \\ 2B\cos(0) - \cos(0) + 0\sin(0) &= 7 \\ 2B - 1 &= 7 \\ 2B &= 8 \\ B &= 4 \end{aligned}$$

STEP 5: Answer

$$y = 4\sin(2t) - t\cos(2t)$$

3. **STEP 1: Standard Form**

$$y'' - \left(\frac{1}{t}\right) y' + \left(1 - \frac{1}{4t^2}\right) y = t^{-\frac{1}{2}}$$

STEP 2: Homogeneous Solution

$$y_0 = At^{-\frac{1}{2}}\cos(t) + Bt^{-\frac{1}{2}}\sin(t)$$

STEP 3: Var of Par

$$y_p(t) = u(t)\, t^{-\frac{1}{2}}\cos(t) + v(t)\, t^{-\frac{1}{2}}\sin(t)$$

$$\begin{bmatrix} t^{-\frac{1}{2}}\cos(t) & t^{-\frac{1}{2}}\sin(t) \\ -\frac{1}{2}t^{-\frac{3}{2}}\cos(t) - t^{-\frac{1}{2}}\sin(t) & -\frac{1}{2}t^{-\frac{3}{2}}\sin(t) + t^{-\frac{1}{2}}\cos(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ t^{-\frac{1}{2}} \end{bmatrix}$$

Denominator:

$$\begin{vmatrix} t^{-\frac{1}{2}}\cos(t) & t^{-\frac{1}{2}}\sin(t) \\ -\frac{1}{2}t^{-\frac{3}{2}}\cos(t) - t^{-\frac{1}{2}}\sin(t) & -\frac{1}{2}t^{-\frac{3}{2}}\sin(t) + t^{-\frac{1}{2}}\cos(t) \end{vmatrix}$$

$$\begin{aligned}
&= t^{-\frac{1}{2}}\cos(t)\left(-\frac{1}{2}t^{-\frac{3}{2}}\sin(t) + t^{-\frac{1}{2}}\cos(t)\right) - t^{-\frac{1}{2}}\sin(t)\left(-\frac{1}{2}t^{-\frac{3}{2}}\cos(t) - t^{-\frac{1}{2}}\sin(t)\right) \\
&= \cancel{-\frac{1}{2}t^{-2}\cos(t)\sin(t)} + t^{-1}\cos^2(t) + \cancel{\frac{1}{2}t^{-2}\sin(t)\cos(t)} + t^{-1}\sin^2(t) \\
&= t^{-1}\left(\cos^2(t) + \sin^2(t)\right) \\
&= t^{-1}
\end{aligned}$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & t^{-\frac{1}{2}}\sin(t) \\ t^{-\frac{1}{2}} & -\frac{1}{2}t^{-\frac{3}{2}}\sin(t) + t^{-\frac{1}{2}}\cos(t) \end{vmatrix}}{t^{-1}} = -t\left(t^{-1}\sin(t)\right) = -\sin(t)$$

$$v'(t) = \frac{\begin{vmatrix} t^{-\frac{1}{2}}\cos(t) & 0 \\ -\frac{1}{2}t^{-\frac{3}{2}}\cos(t) - t^{-\frac{1}{2}}\sin(t) & t^{-\frac{1}{2}} \end{vmatrix}}{t^{-1}} = t\left(t^{-1}\cos(t)\right) = \cos(t)$$

$$u(t) = \int -\sin(t)dt = \cos(t)$$
$$v(t) = \int \cos(t) = \sin(t)$$

STEP 4: Answer

$$\begin{aligned} y_p(t) &= (\cos(t))\, t^{-\frac{1}{2}} \cos(t) + (\sin(t))\, t^{-\frac{1}{2}} \sin(t) \\ &= t^{-\frac{1}{2}} \cos^2(t) + t^{-\frac{1}{2}} \sin^2(t) \\ &= t^{-\frac{1}{2}} \left(\cos^2(t) + \sin^2(t)\right) \\ &= t^{-\frac{1}{2}} \end{aligned}$$

$$y_p(t) = t^{-\frac{1}{2}}$$

4. **STEP 1: Laplace Transforms**

$$\begin{aligned}
\mathcal{L}\left\{y''\right\} + 9\mathcal{L}\left\{y\right\} =& \mathcal{L}\left\{18t \star \delta(t-5)\right\} \\
\left(s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0)\right) 9\mathcal{L}\left\{y\right\} =& \mathcal{L}\left\{18t\right\} \times \mathcal{L}\left\{\delta(t-5)\right\} \\
s^2\mathcal{L}\left\{y\right\} - 1 + 9\mathcal{L}\left\{y\right\} =& \left(\frac{18}{s^2}\right) e^{-5s} \\
\left(s^2+9\right)\mathcal{L}\left\{y\right\} =& 1 + \left(\frac{18}{s^2}\right) e^{-5s} \\
\mathcal{L}\left\{y\right\} =& \frac{1}{s^2+9} + \frac{18}{s^2\left(s^2+9\right)} e^{-5s}
\end{aligned}$$

STEP 2: Partial Fractions

$$\begin{aligned}
\frac{18}{s^2\left(s^2+9\right)} =& \frac{As+B}{s^2} + \frac{Cs+D}{s^2+9} = \frac{(As+B)\left(s^2+9\right) + s^2(Cs+D)}{s^2(s^2+9)} \\
=& \frac{As^3 + 9As + Bs^2 + 9B + Cs^3 + Ds^2}{s^2(s^2+9)} \\
=& \frac{(A+C)s^3 + (B+D)s^2 + 9As + 9B}{s^2(s^2+9)}
\end{aligned}$$

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 9A = 0 \\ 9B = 18 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 2 \\ C = 0 \\ D = -2 \end{cases}$$

$$\frac{18}{s^2\left(s^2+9\right)} = \frac{2}{s^2} - \frac{2}{s^2+9}$$

STEP 3:

$$\begin{aligned}
\mathcal{L}\left\{y\right\} =& \left(\frac{1}{s^2+9}\right) + \left(\frac{2}{s^2} - \frac{2}{s^2+9}\right) e^{-5s} \\
=& \frac{1}{3}\left(\frac{3}{s^2+9}\right) + \left(\frac{2}{s^2} - \frac{2}{3}\left(\frac{3}{s^2+9}\right)\right) e^{-5s} \\
=& \mathcal{L}\left\{\frac{1}{3}\sin(3t)\right\} + \mathcal{L}\left\{2t - \frac{2}{3}\sin(3t)\right\} e^{-5s} \\
=& \mathcal{L}\left\{\frac{1}{3}\sin(3t) + \left(2\left(t-5\right) - \frac{2}{3}\sin(3(t-5))\right) u_5(t)\right\}
\end{aligned}$$

STEP 4: Answer:

$$y = \frac{1}{3}\sin(3t) + \left(2\,(t-5) - \frac{2}{3}\sin(3(t-5))\right)u_5(t)$$

5. **STEP 1:** Complete the square

$$\frac{s+1}{s^2+6s+11} = \frac{s+1}{(s+3)^2-9+11} = \frac{s+1}{(s+3)^2+2}$$
$$= \frac{s+3-2}{(s+3)^2+2} \qquad \text{(Write in terms of } s+3\text{)}$$
$$= \left(\frac{s+3}{(s+3)^2+2}\right) - \left(\frac{2}{(s+3)^2+2}\right)$$

STEP 2: This is a shifted version by -3 units of

$$\frac{s}{s^2+2} - \frac{2}{s^2+2} = \left(\frac{s}{s^2+2}\right) - \sqrt{2}\left(\frac{\sqrt{2}}{s^2+2}\right) = \cos(\sqrt{2}\,t) - \sqrt{2}\sin(\sqrt{2}\,t)$$

Therefore

$$\left(\frac{s+3}{(s+3)^2+2}\right) - \left(\frac{2}{(s+3)^2+2}\right) = \mathcal{L}\left\{e^{-3t}\left(\cos(\sqrt{2}\,t) - \sqrt{2}\sin(\sqrt{2}\,t)\right)\right\}$$

STEP 3: Finally

$$\left(\frac{s+1}{s^2+6s+11}\right)e^{-4s} = \mathcal{L}\left\{e^{-3t}\left(\cos(\sqrt{2}\,t) - \sqrt{2}\sin(\sqrt{2}\,t)\right)\right\}e^{-4s}$$
$$= \mathcal{L}\left\{e^{-3(t-4)}\left(\cos(\sqrt{2}(t-4)) - \sqrt{2}\sin(\sqrt{2}\,(t-4))\right)u_4(t)\right\}$$

STEP 4: Answer:

$$f(t) = e^{-3(t-4)}\left(\cos(\sqrt{2}(t-4)) - \sqrt{2}\sin(\sqrt{2}\,(t-4))\right)u_4(t)$$