APMA 0350 – MIDTERM 2 – SOLUTIONS

1. **STEP 1: Homogeneous Solution**

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow y_0 = A\cos(3t) + B\sin(3t)$$

STEP 2: Particular Solution:

$5\cos(2t) - 10\sin(2t) \rightsquigarrow r = \pm 2i$ so there is no resonance

$$y_p = A\cos(2t) + B\sin(2t)$$

Plug into the ODE:

$$\begin{aligned}(y_p)' &= -2A\sin(2t) + 2B\cos(2t) \\ (y_p)'' &= -4A\cos(2t) - 4B\sin(2t)\end{aligned}$$

$$\begin{aligned}(y_p)'' + 9y_p &= 5\cos(2t) - 10\sin(2t) \\ -4A\cos(2t) - 4B\sin(2t) + 9\left(A\cos(2t) + B\sin(2t)\right) &= 5\cos(2t) - 10\sin(2t) \\ 5A\cos(2t) + 5B\sin(2t) &= 5\cos(2t) - 10\sin(2t)\end{aligned}$$

$$\begin{cases} 5A = 5 \\ 5B = -10 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2 \end{cases}$$

$$y_p = \cos(2t) - 2\sin(2t)$$

STEP 3: General Solution

$$y = y_0 + y_p = A\cos(3t) + B\sin(3t) + \cos(2t) - 2\sin(2t)$$

2. STEP 0: Standard Form ✓

STEP 1: Homogeneous Solution

Aux $r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2 \text{ or } r = 3$

$$y_0 = Ae^{2t} + Be^{3t}$$

STEP 2: Var of Par

$$y_p(t) = u(t)e^{2t} + v(t)e^{3t}$$

$$\begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2t} \end{bmatrix}$$

Denominator:

$$\begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{2t}3e^{3t} - e^{3t}2e^{2t} = 3e^{5t} - 2e^{5t} = e^{5t}$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix}}{e^{5t}} = \frac{0 - 2e^{5t}}{e^{5t}} = -2$$

$$v'(t) = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & 2e^{2t} \end{vmatrix}}{e^{5t}} = \frac{e^{2t}\left(2e^{2t}\right) - 0}{e^{5t}} = \frac{2e^{4t}}{e^{5t}} = 2e^{-t}$$

$$u(t) = \int -2dt = -2t$$

$$v(t) = \int 2e^{-t}dt = -2e^{-t}$$

$$y_p(t) = -2te^{2t} - 2e^{-t}e^{3t} = -2te^{2t} - 2e^{2t}$$

STEP 3:

$$y = y_0 + y_p = Ae^{2t} + Be^{3t} - 2te^{2t} - 2e^{2t} = Ae^{2t} + Be^{3t} - 2te^{2t}$$

3. **STEP 1: Laplace Transforms**

$$\begin{aligned}
\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} =& 4\,\mathcal{L}\{u_5(t)\} \\
\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) - 4\left(s\mathcal{L}\{y\} - y(0)\right) + 4\mathcal{L}\{y\} =& \frac{4}{s}e^{-5s} \\
\left(s^2 - 4s + 4\right)\mathcal{L}\{y\} =& \frac{4}{s}e^{-5s} \\
\mathcal{L}\{y\} =& \frac{4}{s\left(s^2 - 4s + 4\right)}e^{-5s}
\end{aligned}$$

STEP 2: Partial Fractions: From the note, we get

$$\frac{4}{s\left(s^2-4s+4\right)} = \left(\frac{1}{s}\right) - \left(\frac{1}{s-2}\right) + \frac{2}{(s-2)^2}$$

STEP 3:

$$\mathcal{L}\{y\} = \left(\left(\frac{1}{s}\right) - \frac{1}{s-2} + \frac{2}{(s-2)^2}\right)e^{-5s}$$

$\frac{1}{s} = \mathcal{L}\{1\}$ and $\frac{1}{s-2} = \mathcal{L}\left\{e^{2t}\right\}$ and $\frac{2}{(s-2)^2}$ is a shifted version by 2 units of $\frac{2}{s^2} = \mathcal{L}\{2t\}$ so $\frac{2}{(s-2)^2} = \mathcal{L}\left\{e^{2t}2t\right\}$ and hence

$$\begin{aligned}
\mathcal{L}\{y\} =& \mathcal{L}\left\{\left(1 - e^{2t} + 2te^{2t}\right)\right\}e^{-5s} \\
=& \mathcal{L}\left\{\left(1 - e^{2(t-5)} + 2(t-5)e^{2(t-5)}\right)u_5(t)\right\}
\end{aligned}$$

Answer:

$$y = \left(1 - e^{2(t-5)} + 2(t-5)e^{2(t-5)}\right)u_5(t)$$

4. **STEP 1:** Let $I = \int_0^1 x^2 (1-x)^4 \, dx$

Let $f(t) = t^2$ and $g(t) = t^4$ and calculate

$$f \star g = \int_0^t \tau^2 (t-\tau)^4 d\tau$$

STEP 2: u-substitution: Let $u = \frac{\tau}{t}$ then

$$\begin{aligned} f \star g &= \int_0^1 (tu)^2 (t - tu)^4 \, t du \\ &= \int_0^1 (tu)^2 (t(1-u))^4 \, t du \\ &= t^{2+4+1} \underbrace{\int_0^1 u^2 (1-u)^4 du}_{I} \\ &= t^7 (I) \end{aligned}$$

STEP 3:

$$\begin{aligned} \mathcal{L}\{f \star g\} &= \mathcal{L}\left\{t^7 (I)\right\} \\ \mathcal{L}\{f\} \mathcal{L}\{g\} &= I \mathcal{L}\left\{t^7\right\} \\ \mathcal{L}\left\{t^2\right\} \mathcal{L}\left\{t^4\right\} &= I \mathcal{L}\left\{t^7\right\} \\ \left(\frac{2!}{\cancel{s^3}}\right)\left(\frac{4!}{\cancel{s^5}}\right) &= I \left(\frac{7!}{\cancel{s^8}}\right) \\ 2! \, 4! &= I \, (7!) \\ I &= \frac{2! \, 4!}{7!} \end{aligned}$$

STEP 4: Answer

$$I = \int_0^1 x^2 (1-x)^4 dx = \frac{2!4!}{7!}$$