LECTURE: CONFIDENCE INTERVALS (I)

1. Confidence Intervals

Suppose that we're trying to estimate a parameter θ and let $Y_1, \dots, Y_n$ be iid samples from our population.

Definition:

A **confidence interval** uses $Y_1, \dots, Y_n$ to get an interval $[\hat{L}, \hat{U}]$

L and U means lower bound and upper bound, and we use hats to indicate estimators.

Ideally, we would like our interval to have the following properties:

(1) It should contain θ

(2) It should be relatively small

(3) We should be able to calculate the the probability that θ is in our interval

That probability is called the **confidence coefficient**

Definition:

The **confidence coefficient** for $[\hat{L}, \hat{U}]$ is the number $1 - \alpha$ with

$$P(\hat{L} \leq \theta \leq \hat{U}) = 1 - \alpha$$

Note: The reason we use $1-\alpha$ instead of α will be clear once we discuss hypothesis testing. Essentially, α will be the probability of getting a false positive result

2. Confidence Intervals for Large Sample Sizes

In this section, suppose our sample size is large.

For example, for the sample mean $\overline{Y}$ we take $n \geq 30$

Let θ be our parameter and $\hat{\theta}$ be an (unbiased) estimator.

Goal:

Construct a confidence int $[\hat{L}, \hat{U}]$ for $\hat{\theta}$ with confidence coeff $1-\alpha$

Upshot: Since the sample size is large, the Central Limit Theorem says that $\hat{\theta}$ is normally distributed, so it's enough to convert to the standard normal variable Z

STEP 1: Since $\hat{\theta}$ is unbiased by assumption, we have $E(\hat{\theta}) = \theta$

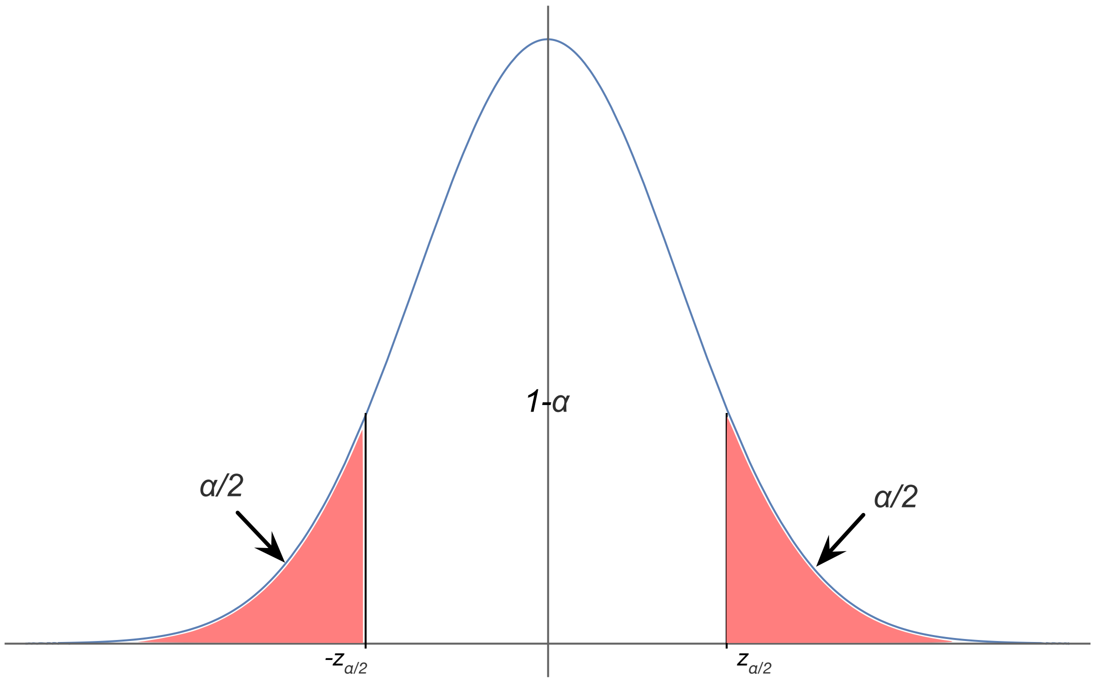
Let $\hat{\sigma}$ be the standard deviation of $\hat{\theta}$. Then:

$$Z = \frac{\hat{\theta} - \theta}{\hat{\sigma}}$$

is (approximately) a standard normal random variable.

STEP 2: Converting to Z, our question is equivalent to finding numbers $-z_{\alpha/2}$ and $z_{\alpha/2}$ such that

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = (1-\alpha)$$



This is done using a $Z-$table

Example: If $(1-\alpha) = 0.95$, then $\alpha/2 = 0.025$. Consulting the Z table, we find that $-z_{\alpha/2} = -1.96$. By symmetry of the standard normal distribution about its mean of 0, $z_{\alpha/2} = 1.96$.

STEP 3: To find $[\hat{L}, \hat{U}]$ just solve for θ

$$P\left(-z_{\alpha/2} \leq \frac{\hat{\theta}-\theta}{\hat{\sigma}} \leq z_{\alpha/2}\right) = 1-\alpha$$

$$P\left(-\hat{\theta} - z_{\alpha/2}\hat{\sigma} \leq -\theta \leq -\hat{\theta} + z_{\alpha/2}\hat{\sigma}\right) = 1-\alpha$$

$$P\left(\hat{\theta} - z_{\alpha/2}\hat{\sigma} \leq \theta \leq \hat{\theta} + z_{\alpha/2}\hat{\sigma}\right) = 1-\alpha$$

STEP 4: **Answer:**

Fact:

Assuming n is large, the $(1-\alpha)$ confidence interval for θ is:

$$[\hat{L}, \hat{U}] = [\hat{\theta} - z_{\alpha/2}\,\hat{\sigma}, \hat{\theta} + z_{\alpha/2}\,\hat{\sigma}]$$

Note: If you also know σ then you can use $\hat{\sigma} = \frac{\sigma}{\sqrt{n}}$ to get

$$[\hat{L}, \hat{U}] = \left[\hat{\theta} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \hat{\theta} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

3. Examples

Example 1:

A ball bearing machine produces ball bearings whose diameters are normally distributed with mean μ mm and standard deviation $\sigma = 0.1$ mm.

To do this, we take $n = 9$ ball bearings and use $\bar{Y} = 10$ mm as our estimator for μ

What is the 90% confidence interval for μ?

This does not require a large sample size because we know Y_i are normal $\Rightarrow \bar{Y}$ is also normal

STEP 1: Since we know σ we use

$$\hat{\sigma} = \frac{\sigma}{\sqrt{n}} = \frac{0.1}{\sqrt{9}} = \frac{0.1}{3} = 0.0333$$

STEP 2: $Z-$table

$$1 - \alpha = 0.9 \Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05$$

Looking at the Z table, we will choose $z_{\alpha/2} = 1.65$ which gives a probability of 0.0495 (ok to choose a probability of 0.505 instead)

STEP 3: Thus our 90% confidence interval for μ is given by:

$$\begin{aligned}[\hat{L}, \hat{U}] &= [\bar{Y} - z_{\alpha/2}\hat{\sigma}, \bar{Y} + z_{\alpha/2}\hat{\sigma}] \\ &= [\bar{Y} - (1.65)(0.0333), \bar{Y} + (1.65)(0.0333)] \\ &= [10 - (1.65)(0.0333), 10 + (1.65)(0.0333)] \\ &= [9.945, 10.055]\end{aligned}$$

Although we cannot say for certain that μ falls within this range, we can say that the probability that it does is 0.90.

Note: In this example, we knew what σ was. If we don't know what σ is, then we would use the sample standard deviation S

Example 2:

The shopping times of $n = 64$ randomly selected customers at a local supermarket were recorded.

The sample mean and sample variance of the 64 shoppers were $\bar{Y} = 33$ minutes and $S^2 = 256$ minutes, respectively.

Find a 98% confidence interval for μ

The parameter of interest is μ and our estimator is $\bar{Y}$

Since the sample is large ($n \geq 30$), by the Central Limit Theorem, we can assume that $\bar{Y}$ is normally distributed.

STEP 1: We do not know σ, we will use the sample variance S^2 hence for $\hat{\sigma}$ we use

$$\hat{\sigma} = \frac{S}{\sqrt{n}} = \frac{\sqrt{256}}{\sqrt{64}} = \frac{16}{8} = 2$$

STEP 2: To find $z_{\alpha/2}$, we look at our Z table

$$(1 - \alpha) = 0.98 \Rightarrow \alpha = 0.02 \Rightarrow \frac{\alpha}{2} = 0.01$$

Looking at our Z table, the closest probability we have to 0.01 is 0.0099 which gives $z_{\alpha/2} = 2.33$

STEP 3: Therefore our 98% confidence interval is

$$\begin{aligned}[\hat{L}, \hat{U}] &= [\bar{Y} - z_{\alpha/2}\hat{\sigma}, \bar{Y} + z_{\alpha/2}\hat{\sigma}] \\ &= [33 - (2.33)(2), 33 + (2.33)(2)] \\ &= [28.34, 37.66]\end{aligned}$$