LECTURE: CONFIDENCE INTERVALS (II)

1. Differences

Here is another example, this time dealing with differences:

Example 1:

Two brands of batteries, Duracell and Energizer, are supposedly guaranteed to last at least 1 year.

In a random sample of 50 Duracell batteries, 12 were found to fail before the 1 year period ended.

In a random sample of 60 Energizer batteries, 12 were also found to fail before the 1 year period ended.

Give a 98% confidence interval for the difference $p_1 - p_2$ between the proportion of failures of the two brands during the 1 year period.

STEP 1: The parameter of interest here is $p = p_1 - p_2$ and

$$\hat{p} = \hat{p_1} - \hat{p_2} = \frac{12}{50} - \frac{12}{60} = 0.24 - 0.2 = 0.04$$

STEP 2: By using the variance of $\frac{Y}{n}$ where Y is binomial, we get

$$(\hat{\sigma})^2 = \text{Var}\left(\hat{p}\right) = \text{Var}\left(\hat{p_1} - \hat{p_2}\right) = \text{Var}\left(\hat{p_1}\right) + \text{Var}\left(\hat{p_2}\right) = \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

$$\hat{\sigma} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

Problem! $\hat{\sigma}$ involves p_1 and p_2, which is what we're trying to estimate!

Solution: Since the population is large, we will $\hat{p}$ in place of p

$$\begin{aligned}\hat{\sigma} &\approx \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \\ &= \sqrt{\frac{(0.24)(0.76)}{50} + \frac{(0.20)(0.80)}{60}} \\ &= 0.0795\end{aligned}$$

STEP 3: In the previous example, we found $z_{\alpha/2} = 2.33$.

STEP 4: Therefore our 98% confidence interval for $p_1 - p_2$ is

$$\begin{aligned}[\hat{L}, \hat{U}] &= [(\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\,\hat{\sigma}, (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\,\hat{\sigma}] \\ &= [0.04 - (2.33)(0.0795), 0.04 + (2.33)(0.0795)] \\ &= [-0.145, 0.225]\end{aligned}$$

2. Confidence Intervals for Small Sample Sizes

Question: What to do when the sample size is small?

Here we can use neither the Central Limit Theorem nor $\sigma^2 \approx S^2$

Assume in this section that the population is normally distributed, since we cannot apply the Central Limit Theorem.

Let $Y_1, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$ where σ is **unknown**

Goal:

Construct a confidence int $[\hat{L}, \hat{U}]$ for μ with confidence coeff $1-\alpha$

This means $P(\hat{L} \leq \mu \leq \hat{U}) = 1 - \alpha$

Upshot: What saves us here is the Student's $t-$distribution

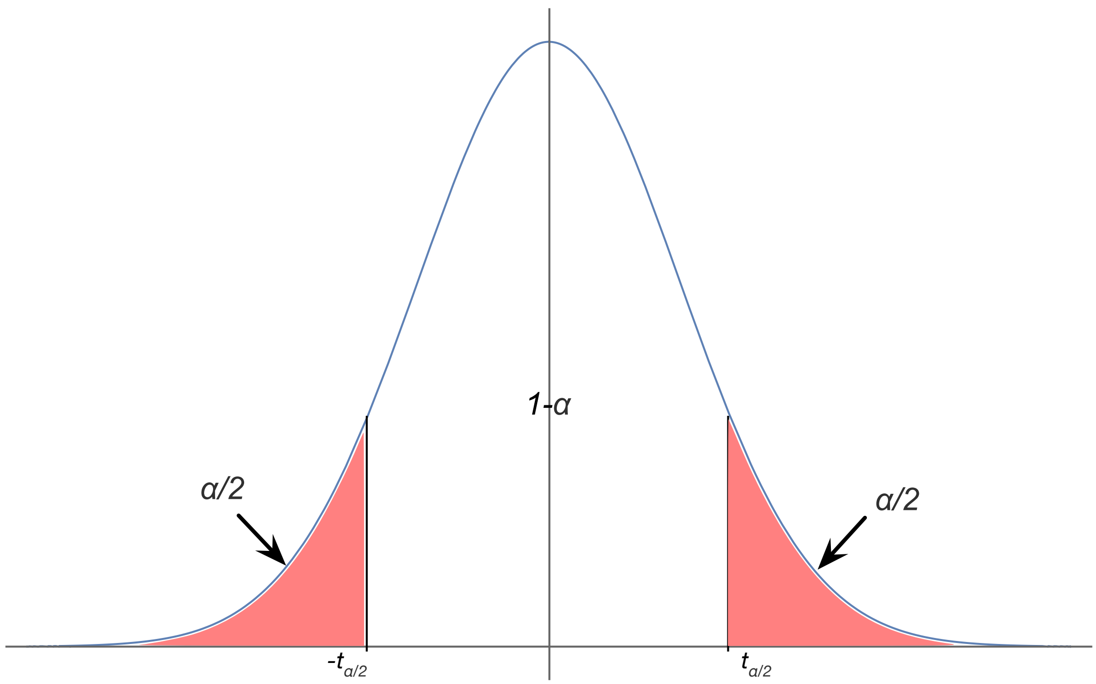
Recall:

If $Y_1, \cdots, Y_n$ are iid $N(\mu, \sigma^2)$ where σ is unknown then

$$T = \frac{\bar{Y} - \mu}{\frac{S}{\sqrt{n}}}$$

Has a **Student's $t-$distribution** with $(n-1)$ degrees of freedom.

So the only difference from before is that we use the t-distribution in place of the standard normal distribution.



STEP 1: Given $(1-\alpha)$, we find $-t_{\alpha/2}$ and $t_{\alpha/2}$ such that

$$P(-t_{\alpha/2} \leq T \leq t_{\alpha/2}) = 1 - \alpha$$

The value $t_{\alpha/2}$ can be found from the t-table with $(n-1)$ df.

STEP 2: Then similar to before, but using $\hat{\sigma} = \frac{S}{\sqrt{n}}$ we get

$$[\hat{L}, \hat{U}] = \left[\bar{Y} - t_{\alpha/2}\left(\frac{S}{\sqrt{n}}\right), \bar{Y} + t_{\alpha/2}\left(\frac{S}{\sqrt{n}}\right)\right]$$

Since the t distribution has thicker tails than the normal distribution, confidence intervals with small sample sizes will be wider than those with large sample sizes.

Example 2:

Suppose you conduct an experiment which involves measuring the launch velocity μ of a model rocket.

Suppose $n = 8$ measurements are taken, and The sample mean $\hat{Y} = 29.59$ m/s, and the sample standard deviation $S = 0.391$ m/s

Find a 95% confidence interval for the true launch velocity of the model rocket.

STEP 1: Since the sample size is small, we will assume that the launch velocities are normally distributed.

Since there are $n = 8$ observations, we use the t distribution with $n - 1 = 7$ df

STEP 2:

$$(1-\alpha) = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$$

From the t-table, we find that $t_{\alpha/2} = t_{0.025} = 2.365$.

STEP 3: Thus our 95% confidence interval is

$$\begin{aligned}[\hat{L}, \hat{U}] &= \left[29.59 - (2.365)\left(\frac{0.391}{\sqrt{8}}\right), 25.959 + (2.365)\left(\frac{0.391}{\sqrt{8}}\right)\right] \\ &= [29.59 - 0.327, 29.59 + 0.327] \\ &= [29.263, 29.917]\end{aligned}$$

3. More Differences

Suppose we have two normal populations and we want to compare the mean $\mu_1 - \mu_2$

Assumption: In this section assume the two populations have the **same** variance $\sigma^2 = \sigma_1^2 = \sigma_2^2$ but σ is unknown.

Our parameter is $\mu_1 - \mu_2$ and our estimator is $\bar{Y}_1 - \bar{Y}_2$ where we sample n_1 people from the first population and n_2 people from the second one.

Goal:

Construct a confidence interval for $\mu_1 - \mu_2$

STEP 1: The main goal is to figure out how to calculate $\hat{\sigma}$

To do this, we first calculate the sample variance S_1^2 for the first sample and the S_2^2 for the second sample

An unbiased estimator for σ^2 can be obtained by combining S_1 and S_2 to obtain the **pooled variance estimator** S_p^2:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)}$$

This is a weighed average of S_1^2 and S_2^2, with the larger sample being given a higher weight.

The weights $(n_1 - 1)$ and $(n_2 - 1)$ are used in place of n_1 and n_2 to get an unbiased estimator

STEP 2:

Fact:

$$T = \frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Has a t distribution with $(n_1 - 1) + (n_2 - 1)$ deg of freedom (df)

STEP 3: Hence, similar to before, a confidence interval for $(\mu_1 - \mu_2)$ is given by:

$$\left[(\bar{Y}_1 - \bar{Y}_2) - \left(t_{\alpha/2}\right) S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, (\bar{Y}_1 - \bar{Y}_2) + \left(t_{\alpha/2}\right) S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right]$$