APMA 1650 – MIDTERM 2 – SOLUTIONS

1. (a)

$$\begin{aligned}\int_{-\infty}^{\infty} f(y)dy &= \int_0^2 c\,(2-y)\,dy \\ &= \int_0^2 2c - cy dy \\ &= \left[2cy - c\left(\frac{y^2}{2}\right)\right]_0^2 \\ &= 2c(2) - c\left(\frac{2^2}{2}\right) \\ &= 4c - 2c \\ &= 2c = 1\end{aligned}$$

$\boxed{c = 1/2}$

(b)

$$\begin{aligned}E(Y) &= \int_{-\infty}^{\infty} y f(y)dy \\ &= \int_0^2 y\left[\frac{1}{2}(2-y)\right]dy \\ &= \int_0^2 y - \frac{1}{2}y^2 dy \\ &= \left[\frac{y^2}{2} - \frac{1}{6}y^3\right]_0^2 \\ &= \frac{4}{2} - \frac{8}{6} \\ &= 2 - \frac{4}{3} \\ &= \frac{2}{3}\end{aligned}$$

$\boxed{E(Y) = \frac{2}{3}}$

2. (a) Let Y be the student's score.

Since we know $E(Y) = 50$ by Markov's inequality with $a = 65$, we have

$$P(Y \geq 65) \leq \frac{E(Y)}{65} = \frac{50}{65}$$

$$P(Y < 65) = 1 - P(Y \geq 65) = 1 - \frac{50}{65} = \frac{15}{65} = \frac{3}{13}$$

Answer: $\boxed{3/13}$

(b) Suppose $\mathrm{Var}(Y) = \sigma^2$

Then by Chebyshev's inequality with $a = 10$ we have

$$P(|Y - E(Y)| \geq 10) \leq \frac{\mathrm{Var}(Y)}{10^2}$$
$$P(|Y - 50| \geq 10) \leq \frac{\sigma^2}{100}$$

$$\begin{aligned} P(40 \leq Y \leq 60) &= P(-10 \leq Y - 50 \leq 10) \\ &= P(|Y - 50| \leq 10) \\ &= 1 - P(|Y - 50| \geq 10) \\ &\geq 1 - \frac{\sigma^2}{100} \end{aligned}$$

In order to ensure that this is ≥ 0.64 it's enough to have

$$\begin{aligned} 1 - \frac{\sigma^2}{100} &\geq 0.64 \\ -\frac{\sigma^2}{100} &\geq -0.36 \\ \frac{\sigma^2}{100} &\leq 0.36 \\ \sigma^2 &\leq 36 \\ \sigma &\leq 6 \end{aligned}$$

Therefore the largest standard deviation allowed is $\boxed{\sigma = 6}$

3. (a)

$$\begin{aligned}E[\hat{\theta}_3] =& E[a\hat{\theta}_1 + (1-a)\hat{\theta}_2] \\ =& aE[\hat{\theta}_1] + (1-a)E[\hat{\theta}_2] \\ =& a\theta + (1-a)\theta \\ =& \theta\end{aligned}$$

Hence $\hat{\theta}_3$ is an unbiased estimator of θ

(b) By the Magic MSE formula, we have

$$\mathrm{MSE}(\hat{\theta}_3) = \left[\mathrm{Bias}(\hat{\theta}_3)\right]^2 + \mathrm{Var}(\hat{\theta}_3)$$

Since $\hat{\theta}_3$ is unbiased we have Bias $\left(\hat{\theta}_3\right) = 0$ hence we just need to find Var $\left(\hat{\theta}_3\right)$.

But since $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, we have

$$\begin{aligned}\mathrm{Var}(\hat{\theta}_3) =& \mathrm{Var}\left(a\hat{\theta}_1 + (1-a)\hat{\theta}_2\right) \\ =& \mathrm{Var}\left(a\hat{\theta}_1\right) + \mathrm{Var}\,(1-a)\hat{\theta}_2 \\ =& a^2\,\mathrm{Var}(\hat{\theta}_1) + (1-a)^2\,\mathrm{Var}(\hat{\theta}_2) \\ =& a^2\sigma_1^2 + (1-a)^2\sigma_2^2\end{aligned}$$

$$\mathrm{MSE}(\hat{\theta}_3) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$

4. Since the sample size is large, by the Central Limit Theorem, $\overline{Y}$ is approximately normal.

Since we don't know σ we use $\hat{\sigma} = \frac{S}{\sqrt{n}} = \frac{3}{10} = 0.3$

$$1 - \alpha = 0.6 \Rightarrow \alpha = 0.4 \Rightarrow \frac{\alpha}{2} = 0.2$$

And from the $z-$table we get $-z_{\alpha/2} = -0.8 \Rightarrow z_{\alpha/2} = 0.8$ and therefore our confidence interval is

$$\begin{aligned}[\hat{L}, \hat{U}] &= \left[\overline{Y} - \left(z_{\alpha/2}\right)\hat{\sigma}, \overline{Y} + \left(z_{\alpha/2}\right)\hat{\sigma}\right] \\ &= [50 - (0.8)(0.3), 50 + (0.8)(0.3)] \\ &= [50 - 0.24, 50 + 0.24] \\ &= [49.76, 50.24]\end{aligned}$$

5. (a)

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1 \int_{x^2}^1 cx\, dydx \\ &= \int_0^1 cx\left(1-x^2\right)dx \\ &= \int_0^1 cx - cx^3 dx \\ &= \left[c\left(\frac{x^2}{2}\right) - c\left(\frac{x^4}{4}\right)\right]_0^1 \\ &= \frac{c}{2} - \frac{c}{4} \\ &= \frac{c}{4} = 1\end{aligned}$$

Hence $\boxed{c = 4}$

(b)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{x^2}^{1} 4xdy = 4x\left(1-x^2\right) = 4x - 4x^3$$

Since $0 \le x \le 1$ we have

$$f_X(x) = \begin{cases} 4x - 4x^3 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$