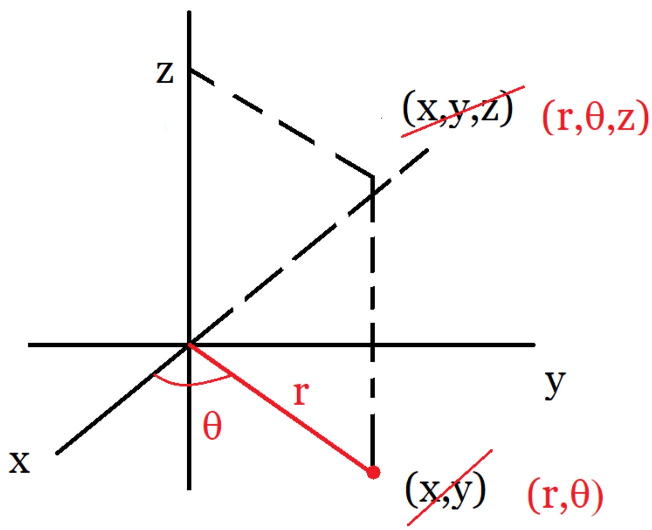
LECTURE 27: CYLINDRICAL COORDINATES

Today: We'll learn about cylindrical coordinates, a "new" coordinate system that helps us evaluate triple integrals. Strictly speaking it's not new because it's just polar coordinates.

1. Cylindrical Coordinates



Cylindrical Coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

(So just do polar on (x, y) and do nothing to z)

Date: Friday, October 29, 2021.

2. Integrals in Cylindrical Coordinates

Video: Cylindrical Integral

Exactly the same thing that we've been doing, just polar coordinates!

Example 1:

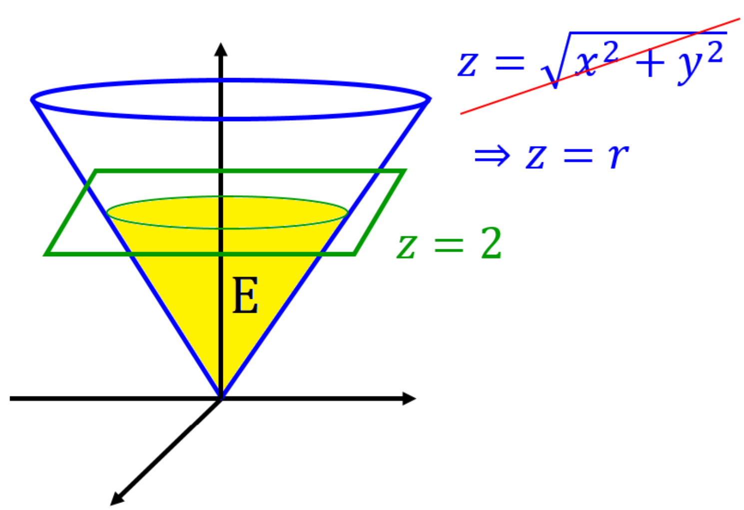
Evaluate the following integral, where E is the region between the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$

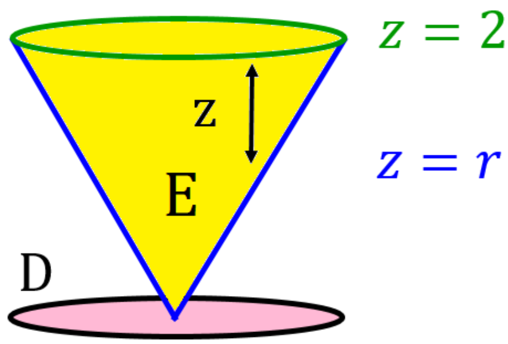
$$\int\int\int_E y^2 dxdydz$$

Rule of Thumb:

Use cylindrical whenever you see $x^2 + y^2$ or a cylinder or a cone

STEP 1: Picture:





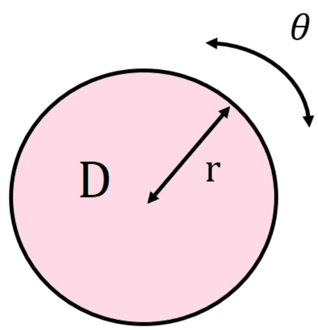
STEP 2: Inequalities:

Note: $z = \sqrt{x^2 + y^2} \Rightarrow z = r$ (much easier)

$$\text{Small } \leq z \leq \text{ Big}$$
$$r \leq z \leq 2$$

Find D:

Intersection: $z = 2$ and $z = r$ gives $r = 2$, so D is a disk of radius 2



$$\begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 2 \end{cases}$$

STEP 3: Integrate

Function: $f(x, y, z) = y^2 = (r\sin(\theta))^2 = r^2 \sin^2(\theta)$

⚠ Do **NOT** forget about the r !!!!

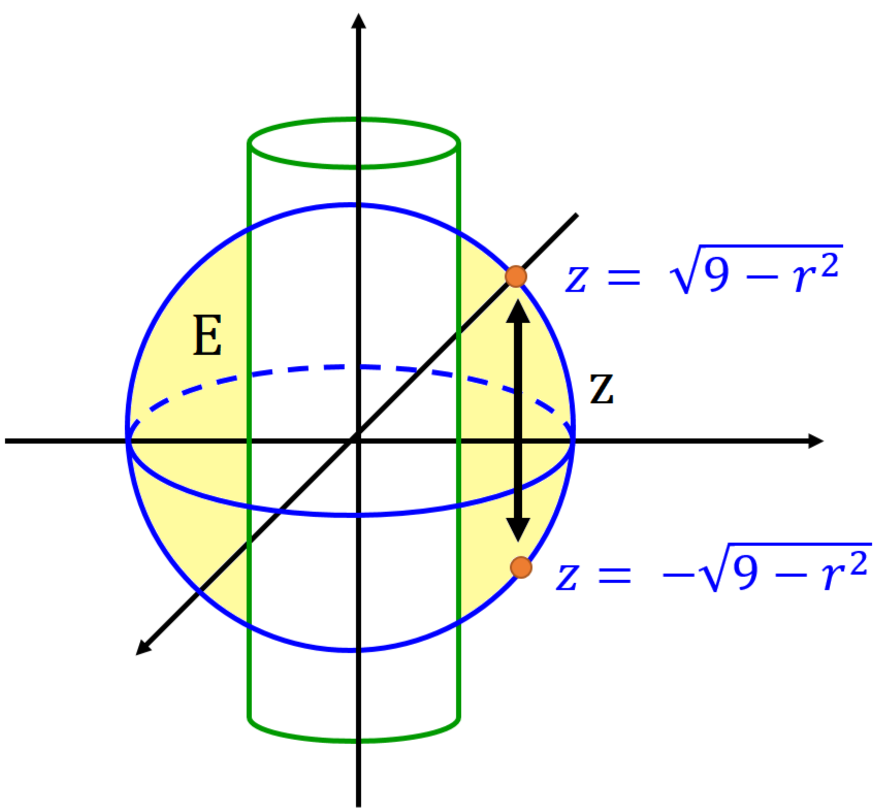
$$\begin{aligned}
&\int\int\int_E y^2 \, dxdydz \\
=&\int_0^{2\pi}\int_0^2\int_r^2 r^2\sin^2(\theta)\, r \, dzdrd\theta \\
=&\int_0^{2\pi}\int_0^2\int_r^2 r^3\sin^2(\theta)\, dzdrd\theta \\
=&\int_0^{2\pi}\int_0^2 (2-r)\, r^3\sin^2(\theta)drd\theta \\
=&\left(\int_0^2 (2-r)r^3 dr\right)\left(\int_0^{2\pi}\sin^2(\theta)d\theta\right) \\
=&\left(\int_0^2 2r^3 - r^4 dr\right)\left(\int_0^{2\pi}\frac{1}{2}-\frac{1}{2}\cos(2\theta)d\theta\right) \\
=&\left[\frac{r^4}{2}-\frac{r^5}{5}\right]_0^2\left[\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\
=&\left(\frac{16}{2}-\frac{32}{5}\right)\left(\frac{2\pi}{2}-0+0-0\right) \\
=&\frac{8\pi}{5}
\end{aligned}$$

Example 2:

Evaluate the following integral, where E is the solid outside the cylinder $x^2 + y^2 = 4$ and inside the sphere $x^2 + y^2 + z^2 = 9$

$$\int\int\int_E \tan^{-1}\left(\frac{y}{x}\right) dxdydz$$

STEP 1: Picture



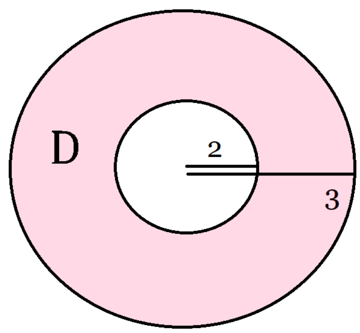
STEP 2: Inequalities:

$$x^2 + y^2 + z^2 = 9 \Rightarrow z^2 = 9 - x^2 - y^2 \Rightarrow z^2 = 9 - r^2 \Rightarrow z = \pm\sqrt{9 - r^2}$$

$$\begin{aligned} \text{Small } &\leq z \leq \text{ Big} \\ -\sqrt{9-r^2} &\leq z \leq \sqrt{9-r^2} \end{aligned}$$

Find D:

If you set $z = 0$ in $x^2+y^2+z^2 = 9$ you get $x^2+y^2 = 9$, which is a circle of radius 3. But since we're focusing on the part outside the cylinder $x^2 + y^2 = 4$, D is actually a ring (annulus):



$$\begin{cases} 2 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

STEP 3: Integrate

Function: $f(x, y, z) = \tan^{-1}\left(\frac{y}{x}\right) = \theta$

$$\begin{aligned} &\int\int\int_E \tan^{-1}\left(\frac{y}{x}\right) dxdydz \\ =& \int_0^{2\pi}\int_2^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} \theta\, r\, dzdrd\theta \\ =& \int_0^{2\pi}\int_2^3 \theta r\left(\sqrt{9-r^2} - \left(-\sqrt{9-r^2}\right)\right) drd\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_2^3 \theta\, r \left(2\sqrt{9-r^2}\right) dr d\theta \\
&= \left(\int_0^{2\pi} \theta d\theta\right) \left(\int_2^3 2r(9-r^2)^{\frac{1}{2}} dr\right) \qquad (u = 9 - r^2) \\
&= \left[\frac{\theta^2}{2}\right]_0^{2\pi} \left[-\frac{2}{3}(9-r^2)^{\frac{3}{2}}\right]_{r=2}^{r=3} \\
&= \frac{4\pi^2}{2}\left(-\frac{2}{3}(9-9)^{\frac{3}{2}} + \frac{2}{3}(9-4)^{\frac{3}{2}}\right) \\
&= 2\pi^2\left(\frac{2}{3}5\sqrt{5}\right) \\
&= \left(\frac{20\sqrt{5}}{3}\right)\pi^2
\end{aligned}
$$

3. Other Directions

Video: Integral over a Princess Cake

Just like last time, our region can sometimes face another direction, like the $x-$direction or the $y-$direction

Example 3:

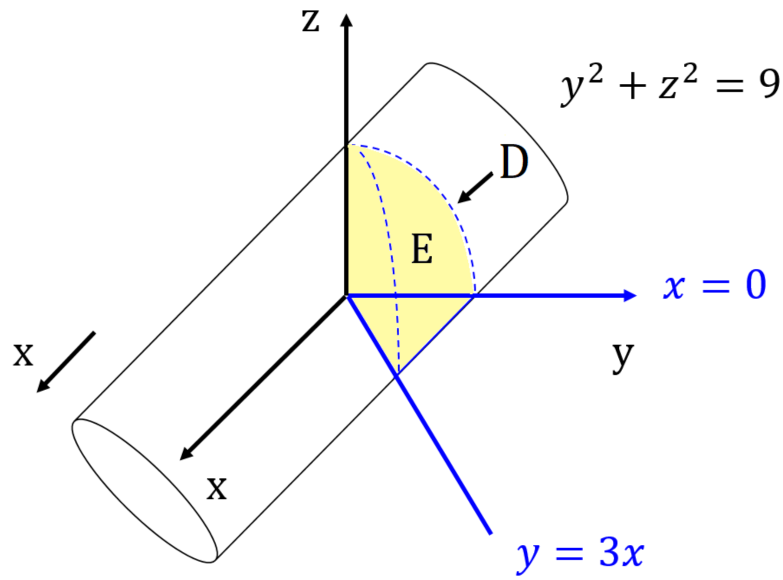
Evaluate the following integral, where E is the solid enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $y = 3x$ in the first octant

$$\int\int\int_E \sqrt{y^2 + z^2}\, dx dy dz$$

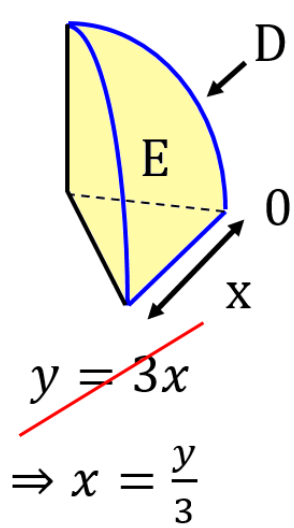
STEP 1: Picture:

$y^2 + z^2 = 4$ is a cylinder in the x-direction

To draw this, start with the cylinder $y^2 + z^2 = 9$, and then cut a wedge along the lines $x = 0$ and $y = 3x$.



Re-Draw:



Here the region is in the $x-$direction.

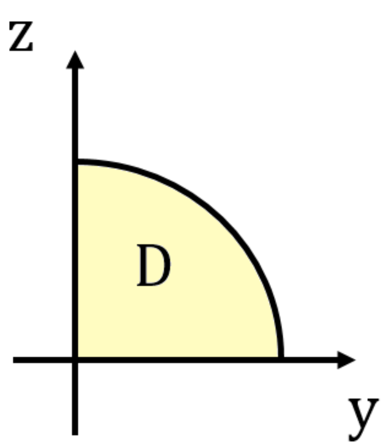
STEP 2: Inequalities:

Note that $y = 3x \Rightarrow x = \frac{y}{3}$

$$\text{Back } \leq x \leq \text{ Front}$$
$$0 \leq x \leq \frac{y}{3}$$

STEP 3: Find D

Here D is the shadow in the back of E, which here is a disk of radius 3 in y and z (because of $y^2 + z^2 = 9$)



$$0 \leq r \leq 3$$
$$0 \leq \theta \leq \frac{\pi}{2}$$

Notice in particular here that $y = r\cos(\theta)$ and $z = r\sin(\theta)$

Therefore our inequalities are

$$\begin{cases} 0 \leq x \leq \dfrac{y}{3} = \dfrac{r\cos(\theta)}{3} \\ 0 \leq r \leq 3 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \end{cases}$$

STEP 4: Integrate

Note: $\sqrt{y^2+z^2} = r$ here.

$$\begin{aligned}
&\int\int\int_E \sqrt{y^2+z^2}\, dxdydz \\
&\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{\frac{r}{3}\cos(\theta)} r\, r dxdrd\theta \\
&\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{\frac{r}{3}\cos(\theta)} r^2\, dxdrd\theta \\
=&\int_0^{\frac{\pi}{2}}\int_0^3 r^2\left(\frac{r}{3}\cos(\theta)\right) drd\theta \\
=&\int_0^{\frac{\pi}{2}}\int_0^3 \frac{r^3}{3}\cos(\theta)drd\theta \\
=&\left(\int_0^3 \frac{r^3}{3}dr\right)\left(\int_0^{\frac{\pi}{2}} \cos(\theta)d\theta\right) \\
=&\left[\frac{r^4}{12}\right]_0^3 [\sin(\theta)]_0^{\frac{\pi}{2}} \\
=&\left(\frac{81}{12}\right)(1) \\
=&\frac{27}{4}
\end{aligned}$$

4. Integral over SIMS

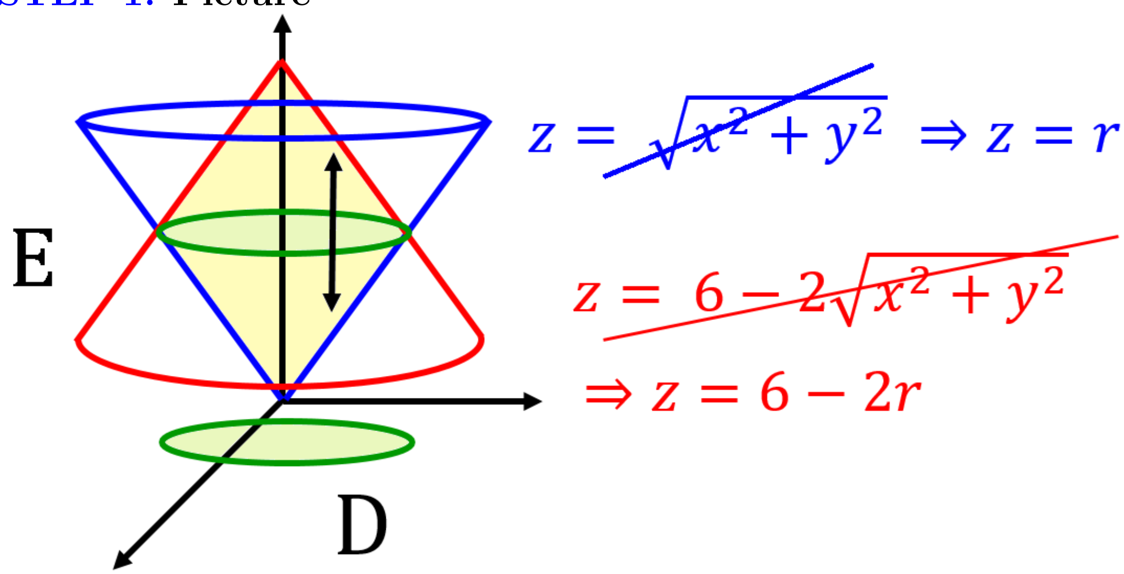
Video: Integral over SIMS

Example 4: (extra practice)

Find the volume of the region E enclosed by the cones

$$z = \sqrt{x^2 + y^2} \text{ and } z = 6 - 2\sqrt{x^2 + y^2}$$

STEP 1: Picture



STEP 2: Inequalities:

$$\text{Small} \leq z \leq \text{Big}$$
$$\sqrt{x^2 + y^2} \leq z \leq 6 - 2\sqrt{x^2 + y^2}$$
$$r \leq z \leq 6 - 2r$$

STEP 3: Find D: Intersection:

$$\begin{aligned}\sqrt{x^2+y^2} =& 6-2\sqrt{x^2+y^2}\\ r =& 6-2r\\ 3r =& 6\\ r =& 2\end{aligned}$$

Hence D is a disk of radius 2, so

$$\begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 6-2r \end{cases}$$

STEP 3: Integrate:

$$\begin{aligned}\mathrm{Vol}(E) =& \int\int\int_E 1\, dxdydz\\ =& \int_0^{2\pi}\int_0^2\int_r^{6-2r} r\, dzdrd\theta\\ =& 2\pi\int_0^2 r\left(6-2r-r\right) dr\\ =& 2\pi\int_0^2 6r-3r^2 dr\\ =& 2\pi\left[3r^2-r^3\right]_0^2\\ =& 2\pi\left(3(2)^2-2^3\right)\\ =& 2\pi\left(12-8\right)\\ =& 2\pi(4)\\ =& 8\pi\end{aligned}$$

5. More Practice

Example 5:

Evaluate the following integral:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} z \, dz dy dx$$

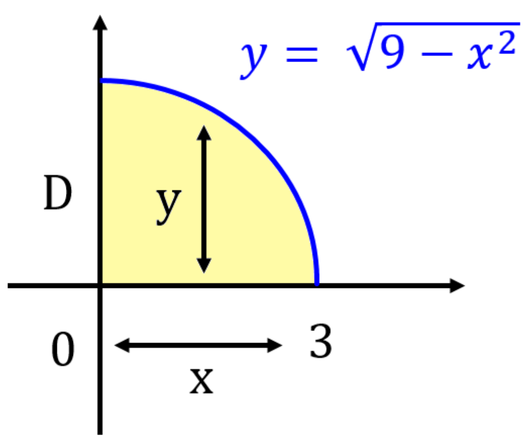
STEP 1: Inequalities:

$$\begin{cases} 0 \leq z \leq 9 - x^2 - y^2 \\ 0 \leq y \leq \sqrt{9 - x^2} \\ 0 \leq x \leq 3 \end{cases}$$

STEP 2: Find D

$y = \sqrt{9 - x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$ (circle of radius 3)

The inequalities for x and y tell us that D is a quarter circle of radius 3 in the first quadrant.



This gives us

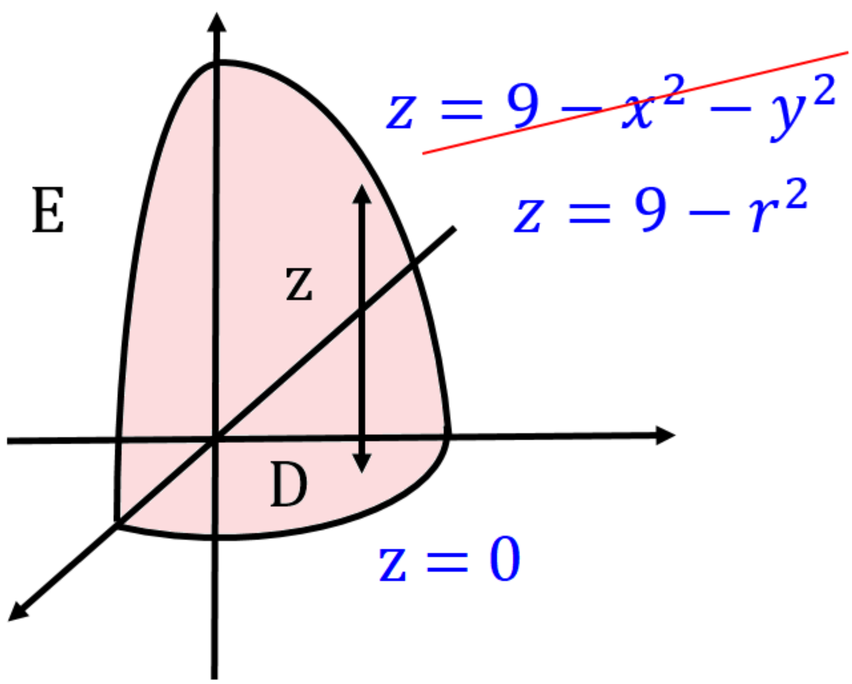
$$0 \leq r \leq 3$$
$$0 \leq \theta \leq \frac{\pi}{2}$$

STEP 3: Picture

Finally focus on z:

$$0 \leq z \leq 9 - x^2 - y^2$$

So z is between the plane $z = 0$ and the paraboloid $z = 9 - x^2 - y^2 = 9 - r^2$, and therefore our solid looks like this:



Inequalities:

$$\begin{cases} 0 \leq r \leq 3 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 \leq z \leq 9 - r^2 \end{cases}$$

STEP 4: Integrate

$$\begin{aligned}
&\int\int\int_E z \, dxdydz \\
&\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{9-r^2} z\, r \, dzdrd\theta \\
=&\frac{\pi}{2}\int_0^3 \left[\left(\frac{z^2}{2}\right) r\right]_{z=0}^{z=9-r^2} dr \\
=&\frac{\pi}{2}\int_0^3 \frac{1}{2}\left(9-r^2\right)^2 r dr \\
=&\frac{\pi}{4}\int_0^3 81r - 18r^3 + r^5 dr \\
=&\frac{\pi}{4}\left[\frac{81}{2}r^2 - \frac{9}{2}r^4 + \frac{1}{6}r^6\right]_0^3 \\
=&\frac{243\pi}{8}
\end{aligned}$$