LECTURE: NONLINEAR SYSTEMS

1. Classification of equilibrium points

Example 1: (continued)

Classify the equilibrium points $(0,0)$ and $(2,1)$ of

$$\begin{cases} x' = -\,x + xy \\ y' = -\,8y + 4xy \end{cases}$$

Case 1: $(0,0)$

STEP 1: Find $\nabla F(0,0)$

$$\nabla F(x,y) = \begin{bmatrix} \frac{\partial(-x+xy)}{\partial x} & \frac{\partial(-x+xy)}{\partial y} \\ \frac{\partial(-8y+4xy)}{\partial x} & \frac{\partial(-8y+4xy)}{\partial y} \end{bmatrix} = \begin{bmatrix} -1+y & x \\ 4y & -8+4x \end{bmatrix}$$

$$\nabla F(0,0) = \begin{bmatrix} -1+0 & 0 \\ 4(0) & -8+4(0) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = A$$

STEP 2: Look at the eigenvalues of $A = \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$ which are

$$\lambda = -1 < 0 \text{ and } \lambda = -8 < 0$$

Summary: Classification

Let (a, b) be an equilibrium solution and let $A = \nabla F(a, b)$

If the eigenvalues of A are

(1) All negative (or negative real part) then (a, b) is **stable**

(2) All positive (or positive real part) then (a, b) is **unstable**

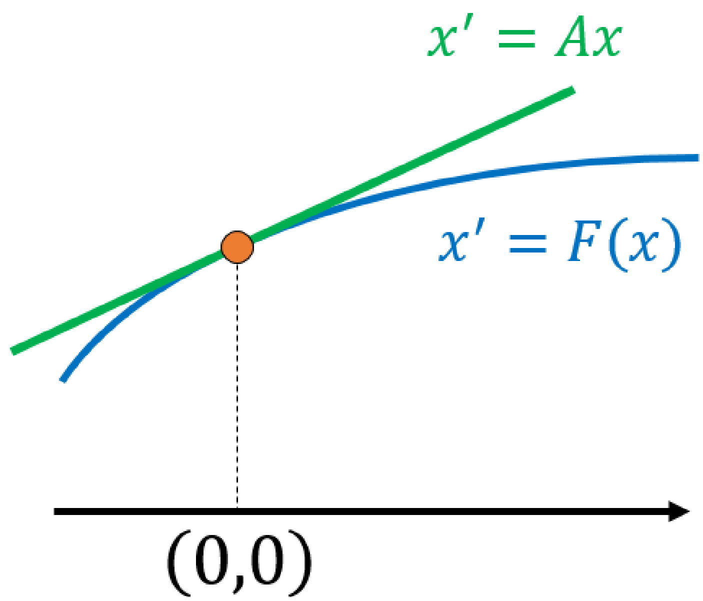
(3) Positive and Negative, then (a, b) is **a saddle**

Therefore $(0, 0)$ is **stable**

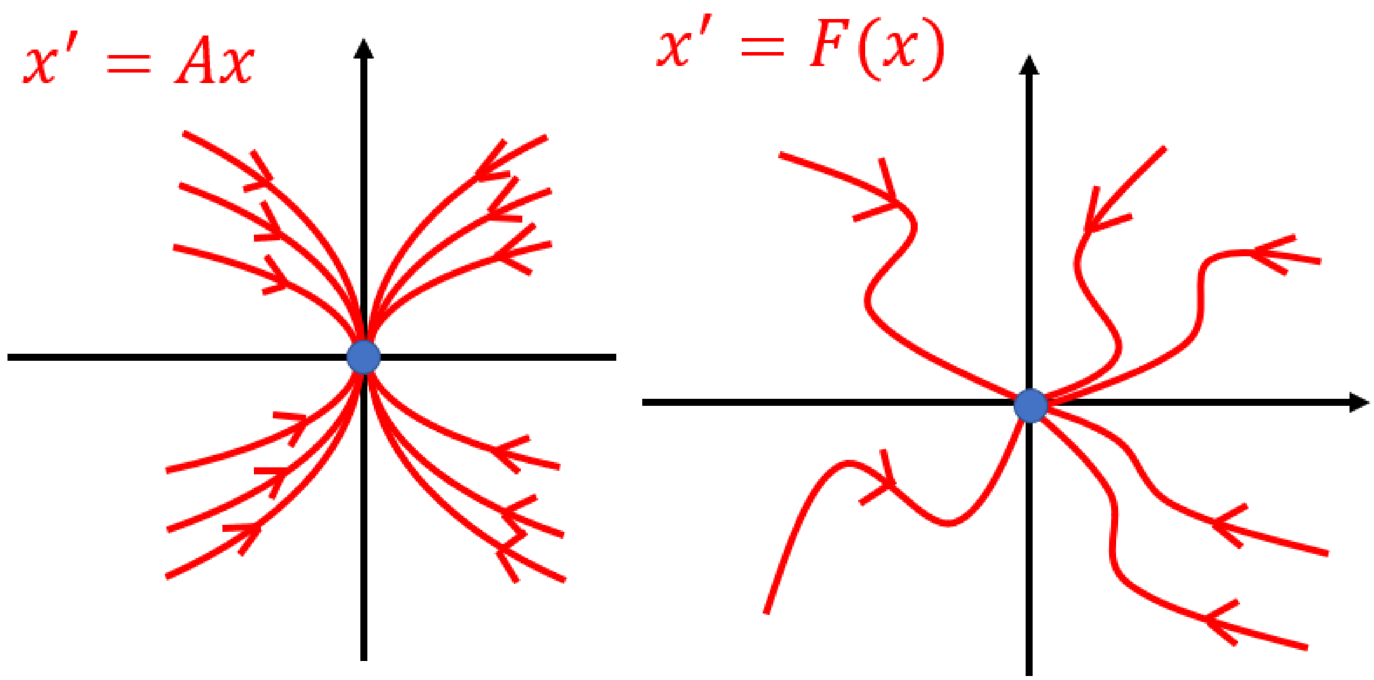
Why true? Here solutions to $\mathbf{x}' = A\mathbf{x}$ go to $(0, 0)$ as $t \to \infty$ because solutions are of the form

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} \star \\ \star \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} \star \\ \star \end{bmatrix}$$

By definition of a derivative, solutions to $\mathbf{x}' = F(\mathbf{x})$ are close to solutions of $\mathbf{x}' = A\mathbf{x}$ as illustrated in the picture below



Hence solutions to $\mathbf{x}' = F(\mathbf{x})$ go towards $(0,0)$ as well:

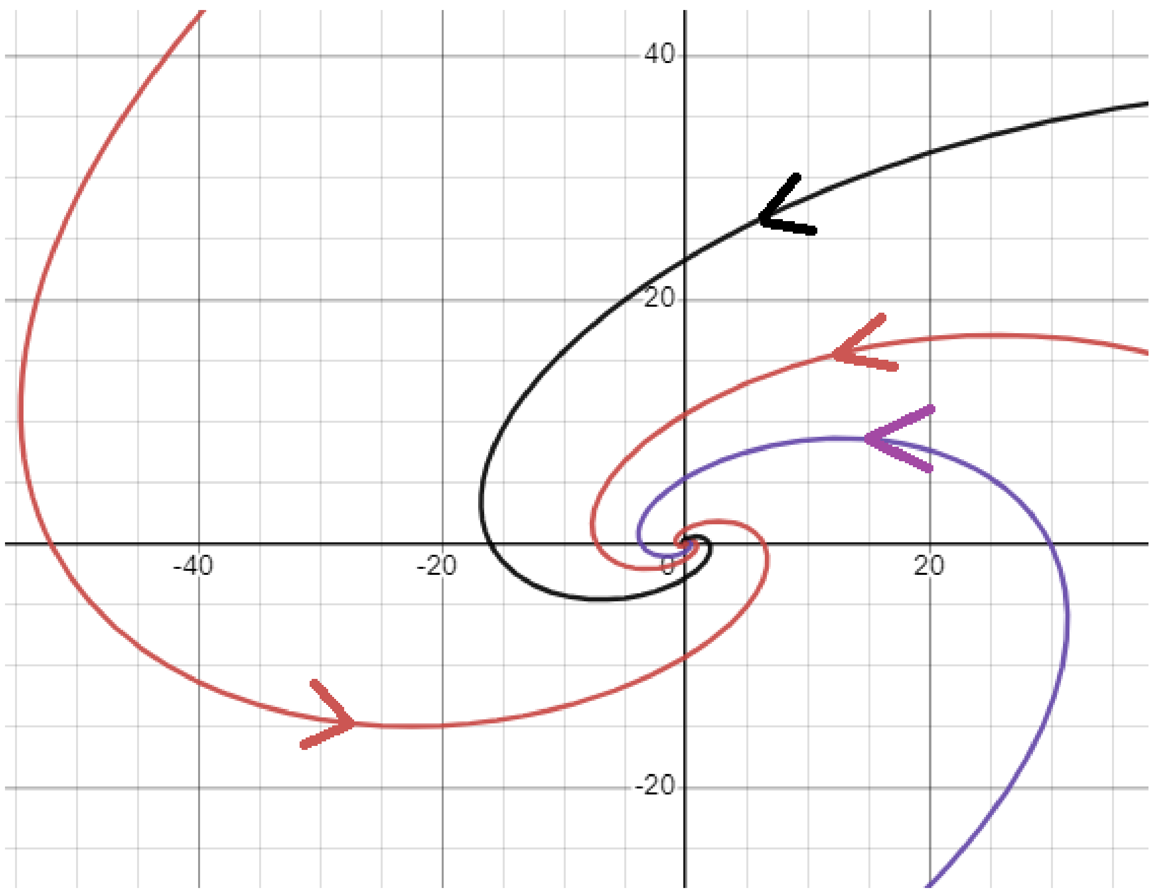


Remarks:

(1) An example of a saddle is given below

(2) If one of the eigenvalues is 0 or purely imaginary like $\lambda = \pm 3i$ this is called **degenerate** and will be dealt on a case-by-case basis.

(3) Here is an example with $\lambda = -2 \pm 3i$ where the real part of λ is negative. $(0,0)$ is still stable: Solutions near $(0,0)$ move towards it



Case 2: $(2,1)$

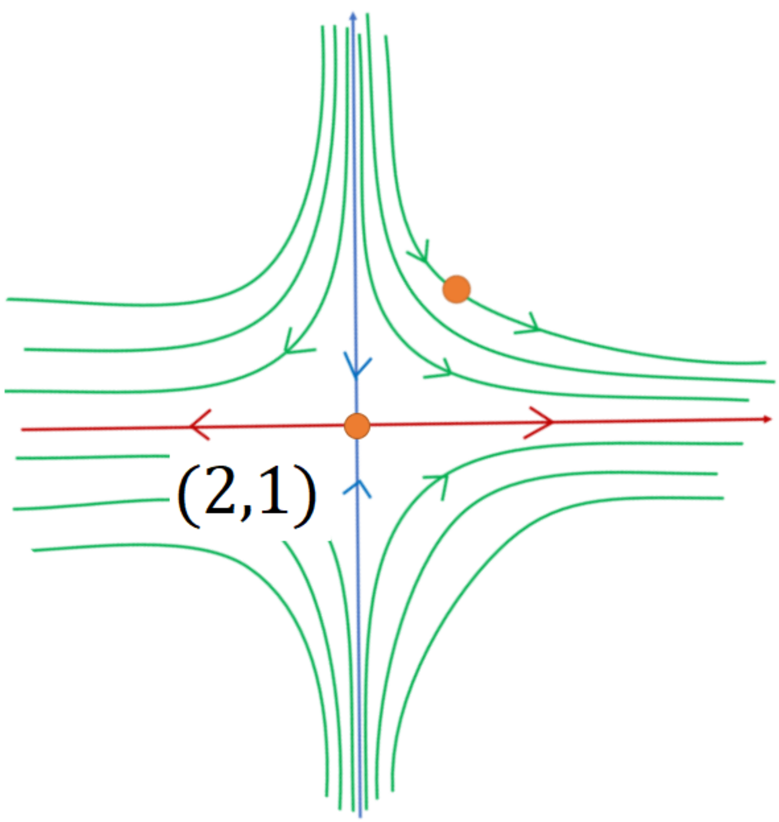
$$\nabla F(x,y) = \begin{bmatrix} -1+y & x \\ 4y & -8+4x \end{bmatrix}$$

$$\nabla F(2,1) = \begin{bmatrix} -1+1 & 2 \\ 4(1) & -8+4(2) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} = A$$

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 - 8 = 0 \Rightarrow \lambda = \pm\sqrt{8}$$

Since one eigenvalue is positive and one is negative $(2,1)$ is a **saddle**



2. Application 1: Competing Species

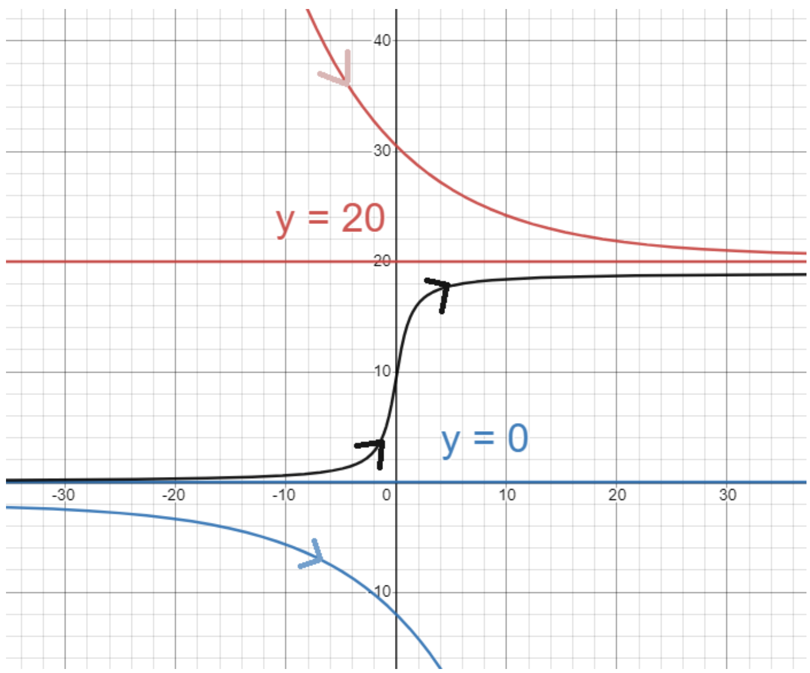
Recall: In our population adventure, discussed the logistic equation

Logistic Equation

$$y' = 3y\left(1 - \frac{y}{20}\right)$$

3 is the growth rate and 20 is the limiting population/carrying capacity.

We saw that this was a pretty good model for population growth.



Question: What if you have **two** animal species, say rabbits and sheep, that are living together?

Note: This is **NOT** a bunnies vs. fox model, because the sheep do not eat the bunnies. That said, both populations still compete with each other for limited resources. Think for instance where the supply of hay is limited and they both need to eat it for survival.

Unknowns:

$$\begin{cases} x(t) = \text{Population of Rabbits} \\ y(t) = \text{Population of Sheep} \end{cases}$$

One simple way of modeling $x(t)$ and $y(t)$ is to have two separate logistic equations:

Logistic Model:

$$\begin{cases} x' =3x\left(1-\dfrac{x}{\frac{3}{2}}\right) \\ y' =2y\left(1-\dfrac{y}{2}\right) \end{cases} \Rightarrow \begin{cases} x' =3x-2x^2 \\ y' =2y-y^2 \end{cases}$$

This is **BAAAAAAD** because it doesn't take into account the *interaction* between the bunnies and the sheep! In particular, you should expect that, the more bunnies there are, the less hay will be available for sheep, which slows the population of sheep.

Our Model:

$$\begin{cases} x' = 3x - 2x^2 - xy \\ y' = 2y - y^2 - xy \end{cases}$$

This makes sense, because if there are no bunnies, then $x = 0$ and so $-xy = 0$ so the sheep will grow according to the logistic model $y' = 2y - y^2$ The $-$ sign accounts for the competition between the species.

Note: You can generalize this by putting different constants in front of the $-xy$ like $-2xy$ for x' and $-3xy$ for y'

3. Equilibria

Example 2:

Find the equilibrium points of the system above

$$\begin{cases} x' = 3x - 2x^2 - xy = 0 \\ y' = 2y - y^2 - xy = 0 \end{cases} \Rightarrow \begin{cases} x\,(3 - 2x - y) = 0 \\ y\,(2 - y - x) = 0 \end{cases}$$

Case 1: $x = 0$

Then either $y = 0$ which gives $(0, 0)$

Or $2 - y - x = 0 \Rightarrow 2 - y - 0 = 0 \Rightarrow y = 2$ which gives $(0, 2)$

Case 2: $3 - 2x - y = 0$

Either $y = 0$, so $3 - 2x - y = 0 \Rightarrow 3 - 2x - 0 = 0 \Rightarrow x = \frac{3}{2}$ so $\left(\frac{3}{2}, 0\right)$

Or $2 - y - x = 0$, and so we have to solve the system

$$\begin{cases} 3 - 2x - y = 0 \\ 2 - y - x = 0 \end{cases}$$

Use $2 - y - x = 0 \Rightarrow y = 2 - x$ and hence

$$3 - 2x - (2 - x) = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

And $y = 2 - x = 2 - 1 = 1$ which gives $(1, 1)$

$$\textbf{Answer: } (0,0), (0,2), \left(\frac{3}{2}, 0\right), (1,1)$$

4. Classification

Example 3:

Classify the equilibria above

$$\nabla F(x,y) = \begin{bmatrix} \frac{\partial(3x-2x^2-xy)}{\partial x} & \frac{\partial(3x-2x^2-xy)}{\partial y} \\ \frac{\partial(2y-y^2-xy)}{\partial x} & \frac{\partial(2y-y^2-xy)}{\partial y} \end{bmatrix} = \begin{bmatrix} 3 - 4x - y & -x \\ -y & 2 - 2y - x \end{bmatrix}$$

Case 1: $(0, 0)$

$$\nabla F(0,0) = \begin{bmatrix} 3 - 0 - 0 & 0 \\ 0 & 2 - 0 - 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues are $\lambda = 3 > 0$ and $\lambda = 2 > 0$, so $(0,0)$ is **unstable**

Case 2: $(0,2)$

$$\nabla F(0,2) = \begin{bmatrix} 3-0-2 & 0 \\ -2 & 2-2(2)-0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

The eigenvalues are $\lambda = 1 > 0$ and $\lambda = -2 < 0$, so $(0,2)$ is a **saddle**

Case 3: $\left(\frac{3}{2}, 0\right)$

$$\nabla F\left(\frac{3}{2}, 0\right) = \begin{bmatrix} 3-4\left(\frac{3}{2}\right)-0 & -\frac{3}{2} \\ -0 & 2-2(0)-\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

The eigenvalues are $\lambda = -3 < 0$ and $\lambda = \frac{1}{2} > 0$ so $\left(\frac{3}{2}, 0\right)$ is a **saddle**

Case 4: $(1,1)$

$$\nabla F(1,1) = \begin{bmatrix} 3-4(1)-1 & -1 \\ -1 & 2-2(1)-1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned} \det(A-\lambda I) = & \begin{vmatrix} -2-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} \\ = & (-2-\lambda)(-1-\lambda) - (-1)(-1) \\ = & 2 + 2\lambda + \lambda + \lambda^2 - 1 \\ = & \lambda^2 + 3\lambda + 1 = 0 \end{aligned}$$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

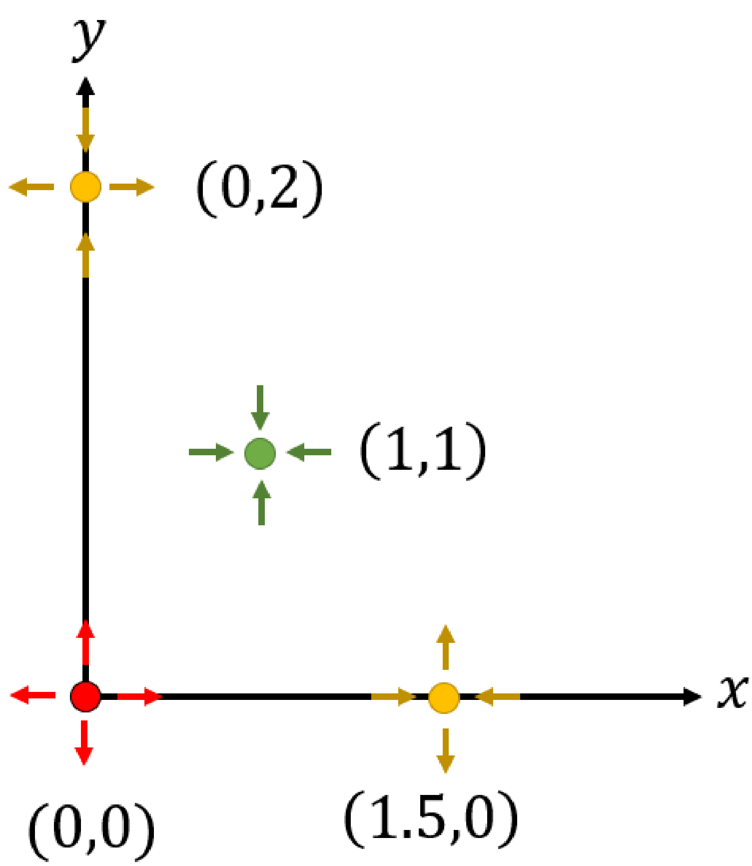
$$\lambda = \frac{-3-\sqrt{5}}{2} < 0 \text{ and } \lambda = \frac{-3+\sqrt{5}}{2} < 0 \text{ hence } (1,1) \text{ is } \textbf{stable}$$

(Here we used $\sqrt{5} < 3$ which follows since $\left(\sqrt{5}\right)^2 < 3^2$, which is $5 < 9$)

Classification:

$$(0,0) \text{ unstable} \quad (0,2) \text{ saddle} \quad \left(\frac{3}{2},0\right) \text{ saddle} \quad (1,1) \text{ stable}$$

In other words, so far we have the following picture:



(The orientation of the saddle points is yet to be confirmed)