LECTURE: ECOLOGY: COMPETING SPECIES

1. Recap

Last time: Studied the dynamics of rabbits and sheep, which compete with each other for limited resources.

$$\begin{cases} x(t) = \text{Population of Rabbits} \\ y(t) = \text{Population of Sheep} \end{cases}$$

Our Model:

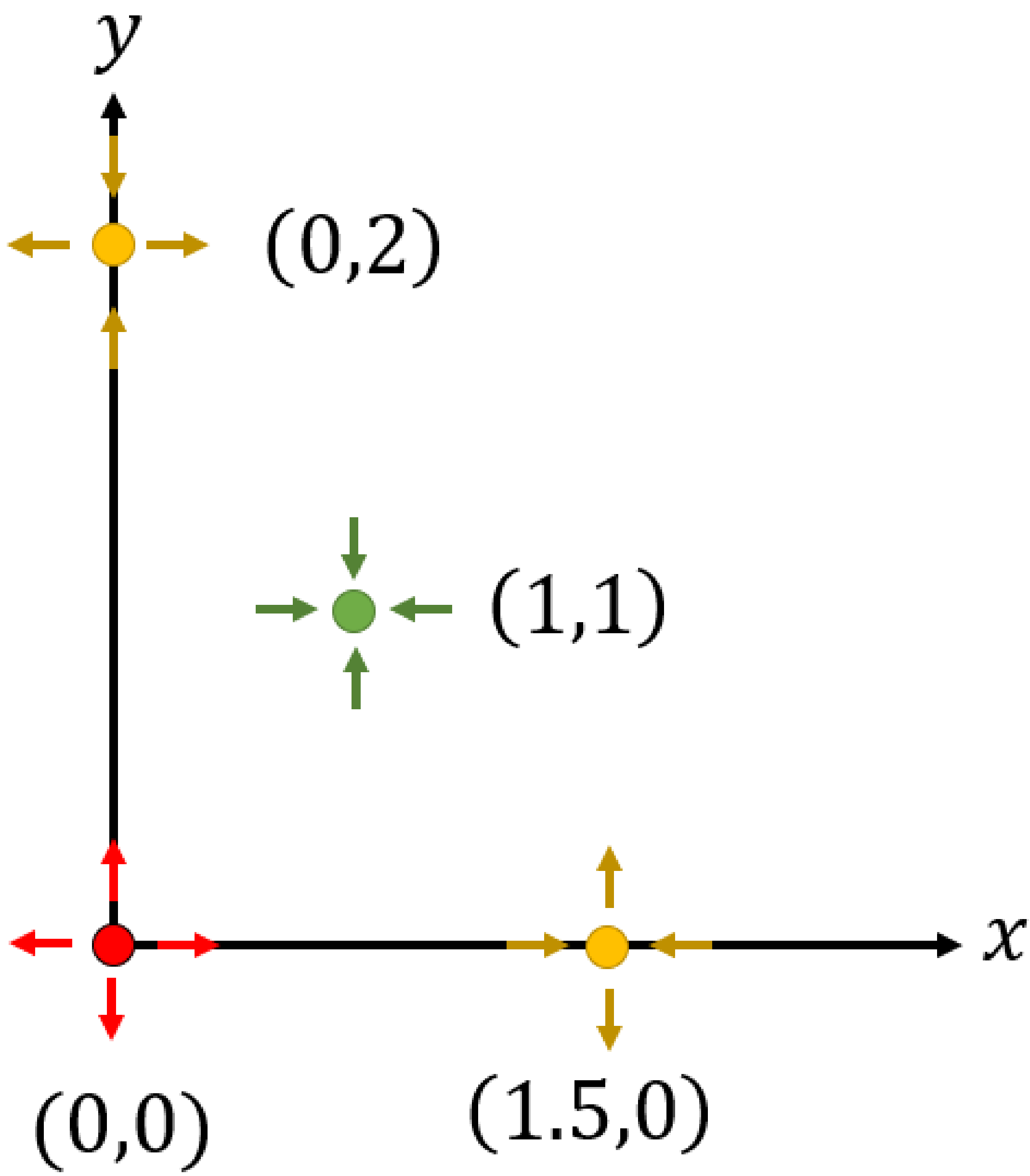
$$\begin{cases} x' = 3x - 2x^2 - xy \\ y' = 2y - y^2 - xy \end{cases}$$

Goal: Will one of the populations go extinct? Will they coexist?

Equilibrium Points:

$$(0,0) \text{ unstable} \quad (0,2) \text{ saddle} \quad \left(\frac{3}{2}, 0\right) \text{ saddle} \quad (1,1) \text{ stable}$$

In other words, so far we have the following picture:



(The orientation of the saddle points is yet to be confirmed)

2. Nullclines

To complete this, let's study the behavior of solutions on the axes $x = 0$, $y = 0$, $x = \frac{3}{2}$ and $y = 2$ (see picture below)

Case 1: $x = 0$ (no rabbits)

Then our system becomes

$$\begin{cases} x' = 3(0) - 2(0)^2 - 0(y) \\ y' = 2y - y^2 - 0(y) \end{cases} \Rightarrow \begin{cases} x' = 0 \\ y' = 2y - y^2 \end{cases} \Rightarrow \begin{cases} x' = 0 \\ y' = y(2 - y) \end{cases}$$

On the part of the axis $x = 0$ where $0 \leq y \leq 2$ (see picture below) we have $y' = y(2 - y) \geq 0$ so y is increasing and the arrows go up.

Interpretation: In the absence of bunnies, the population of sheep is increasing until it reaches the carrying capacity $y = 2$.

Case 2: $y = 0$ (no sheep)

This is similar; in this case the system becomes

$$\begin{cases} x' = 3x - 2x^2 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} x' = x\,(3 - 2x) \\ y' = 0 \end{cases}$$

And we get x is increasing until it reaches the carrying capacity $\frac{3}{2}$

Case 3: $y = 2$ (sheep reached the carrying capacity)

$$\begin{cases} x' = 3x - 2x^2 - x(2) \\ y' = 2(2) - 2^2 - x(2) \end{cases} \Rightarrow \begin{cases} x' = x - 2x^2 \\ y' = -2x \end{cases} \Rightarrow \begin{cases} x' = x(1 - 2x) \\ y' = -2x \end{cases}$$

From the second equation, we have $y' = -2x < 0$ so on the axis $y = 2$, the arrows are pointing **down**

Note: You can get even more info by studying the sign of x' but for our purposes this is enough

Interpretation: If the sheep reach their carrying capacity, their population will start to decrease, due to the competition with the rabbits.

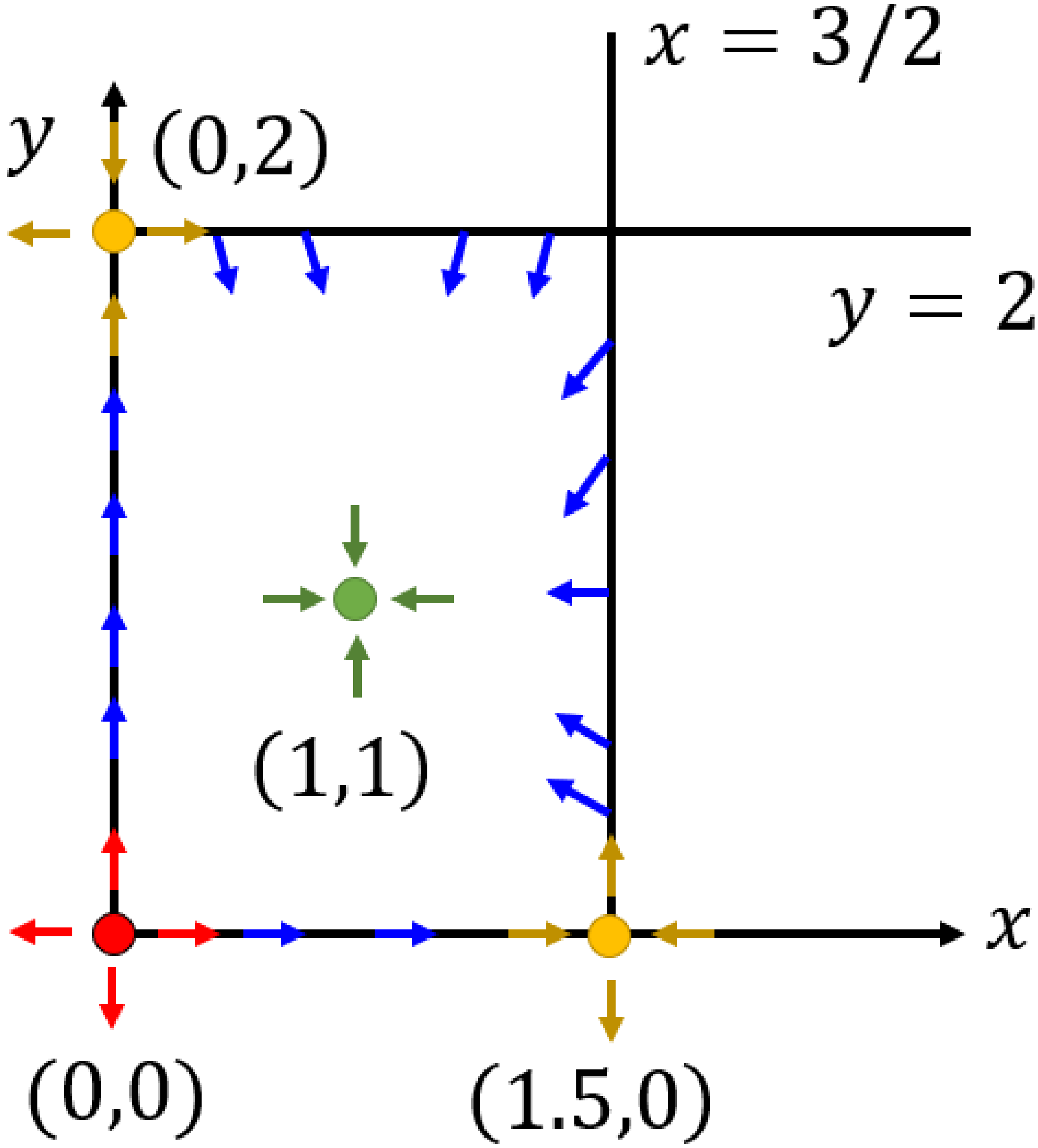
Case 4: $x = \frac{3}{2}$ (rabbits reached the carrying capacity)

$$\begin{cases} x' = 3\left(\frac{3}{2}\right) - 2\left(\frac{3}{2}\right)^2 - \frac{3}{2}y \\ y' = 2y - y^2 - \frac{3}{2}y \end{cases} \Rightarrow \begin{cases} x' = \frac{9}{2} - \frac{9}{2} - \frac{3}{2}y \\ y' = \frac{1}{2}y - y^2 \end{cases} \Rightarrow \begin{cases} x' = -\frac{3}{2}y \\ y' = \frac{1}{2}y - y^2 \end{cases}$$

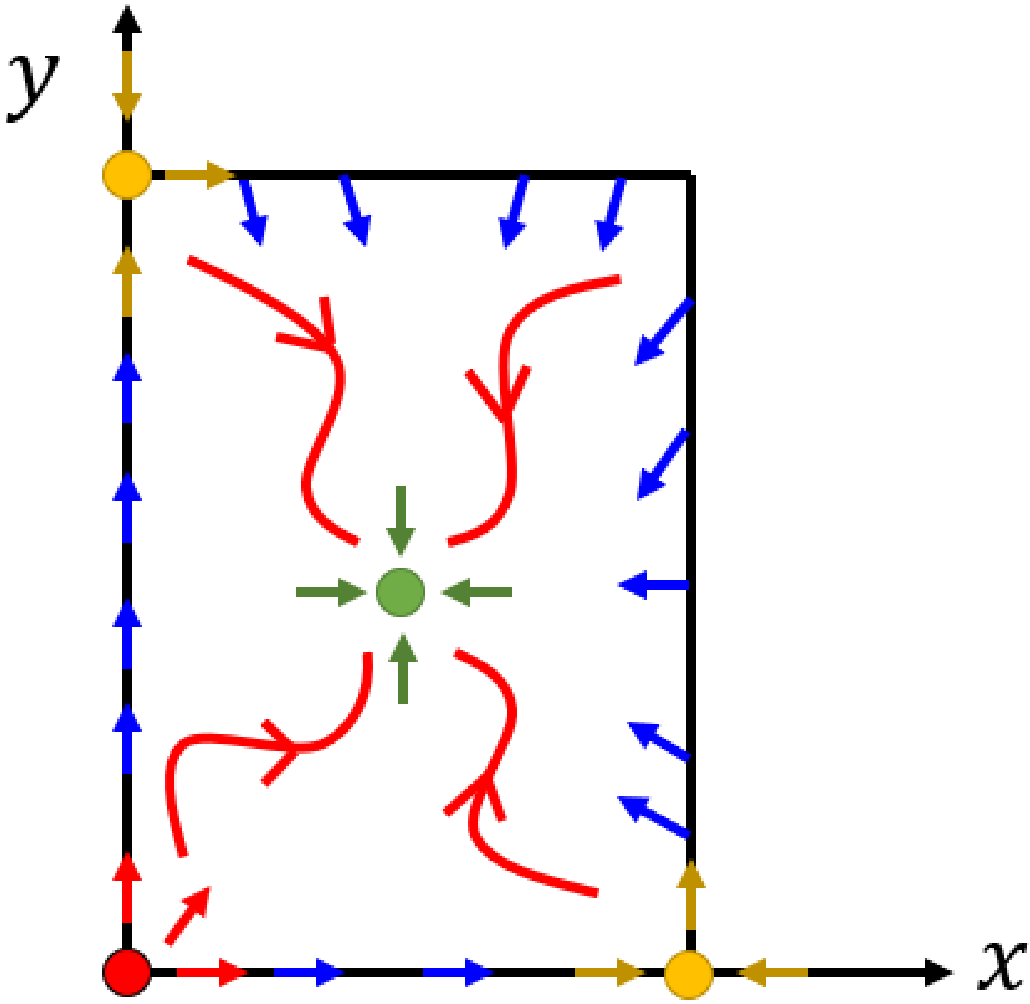
We have $x' = -\frac{3}{2}y < 0$ so on the axis $x = \frac{3}{2}$ the arrows point to the **left**

Note: Curves where $x' = 0$ **or** $y' = 0$ are called **nullclines**. So here $x = 0$ and $y = 0$ are nullclines but $y = 2$ and $x = \frac{3}{2}$ are not

Now we can complete the picture we had at the beginning:



And we can draw some sample solutions simply by following the arrows



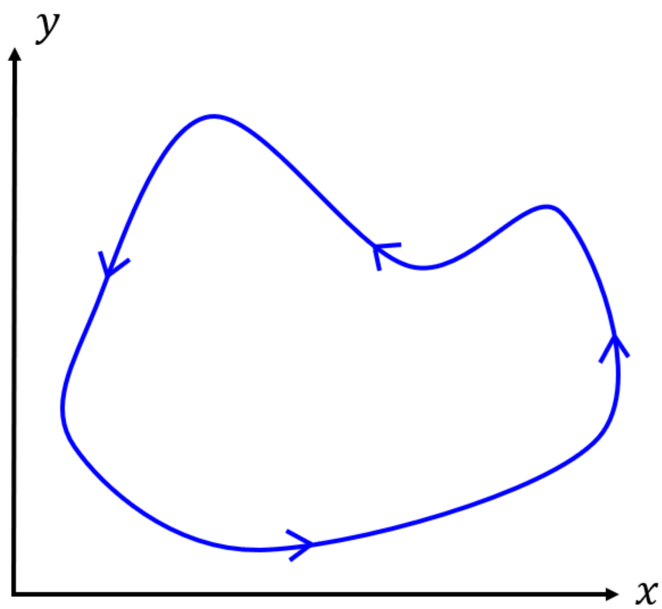
Epic Conclusion:

The two species won't go extinct!! They will happily coexist and eventually reach the equilibrium $(1, 1)$

EXCEPT we need to rule out periodic curves:

3. Periodic Curves (more practice)

What if we have a periodic curve, where the x and y go back-and-forth?



Let's show that this doesn't happen by studying the axes $y = 1$ and $x = 1$ which are the axes where $(1, 1)$ lies on.

Case 1: $y = 1$

$$\begin{cases} x' = 3x - 2x^2 - x(1) \\ y' = 2(1) - 1^2 - x(1) \end{cases} \Rightarrow \begin{cases} x' = 2x - 2x^2 \\ y' = 1 - x \end{cases}$$

Here $y' = 1 - x$ is positive if $x < 1$ and negative if $x > 1$

So on the axis $y = 1$, if $x < 1$, the arrows are pointing up, and if $x > 1$, they are pointing down.

Case 2: $x = 1$

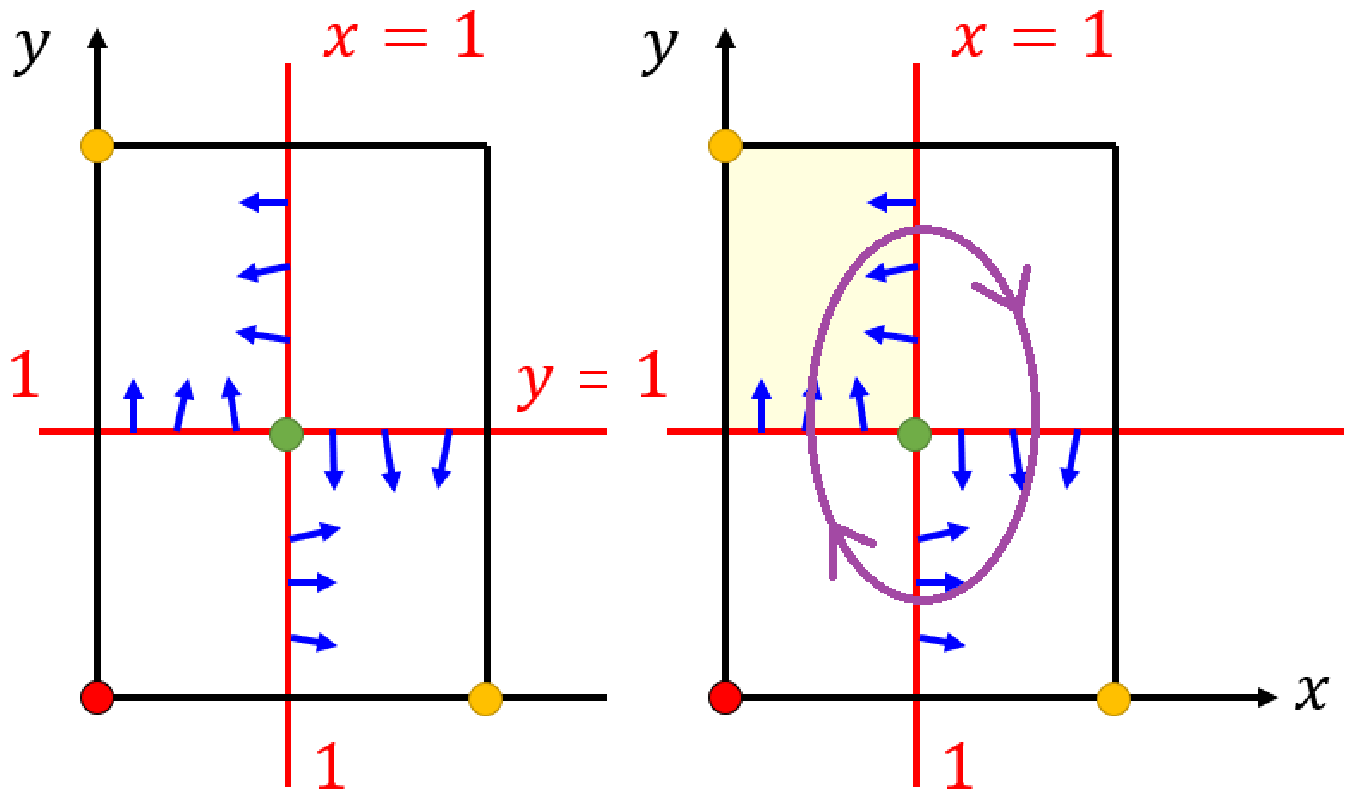
$$\begin{cases} x' = 3(1) - 2(1)^2 - 1y \\ y' = 2y - y^2 - 1y \end{cases} \Rightarrow \begin{cases} x' = 1 - y \\ y' = -y^2 + y \end{cases}$$

Here $x' = 1 - y$ is positive if $y < 1$ and negative if $y > 1$

So on the axis $x = 1$, if $y < 1$, the arrows are pointing to the right, and if $y > 1$, they are pointing to the left

We can therefore complete the picture as below.

Now look at the second quadrant (shaded in yellow) and try to draw a cycle that is consistent with the arrows! This is impossible and tells you that there can't be a periodic curve.



4. Application 2: Epidemiology: SIR models

Goal: ODE model for COVID−19. In particular, we'd like to know

(1) How does COVID-19 spread?

(2) When do outbreaks occur?

(3) How to "flatten the curve?" How to manage outbreaks?

(4) How to prevent outbreaks? How effective *are* vaccines, really?

Suppose you have three groups of people:

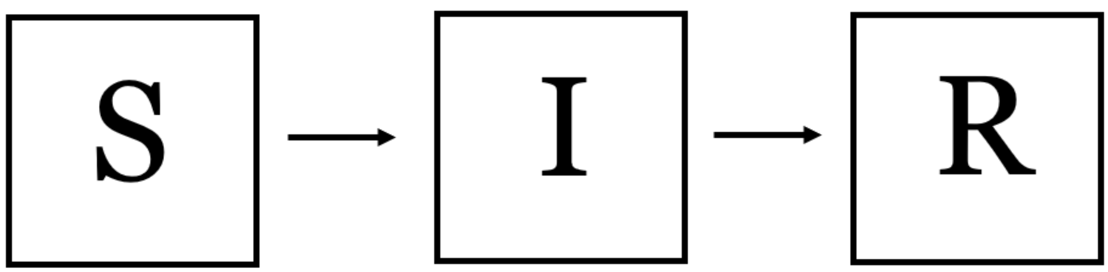
$$\begin{cases} S = \text{susceptible (exposed to COVID)} \\ I = \text{Infected} \\ R = \text{Recovered} \end{cases}$$

Assumptions:

(1) An infected person immediately infects other people

(2) Recovered people don't get reinfected, they are immune

Because of the interactions between the groups, it makes sense to think of this as a compartmental/chemical tank problem:



We just need to figure out the rates between each tank:

$\boxed{I \to R}$ This rate is constant, equal to γ

γ is the percentage of the infected population that recovers per day.

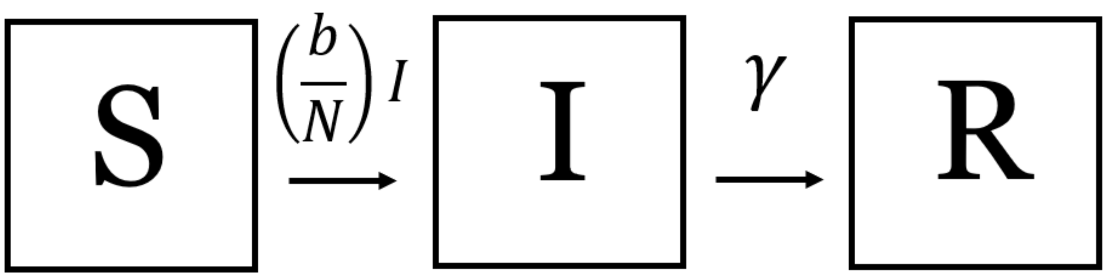
Note: $1/\gamma$ is called the **mean residence time** of the disease. In other words, if $\gamma = 0.25$ then 25% of infected people recover per day so 1 person recovers fully after $1/\gamma = 4$ days

$\boxed{S \to I}$ This rate depends on I: The bigger the number of infected people, the faster the suspected people become infected

Assume the rate $S \to I$ is $\underbrace{\left(\frac{b}{N}\right)}_{\text{Some number}} I$

$$
\begin{aligned}
N =& S + I + R \text{ (the total population)} \\
b =& \text{ number of people a sick person infects per day}
\end{aligned}
$$

Hence we can complete our model:



$$
\begin{aligned}
S'(t) &= \text{In} \ - \ \text{Out} \ = 0 - \left(\frac{b}{N}\right) I(t) \times S(t) \\
I'(t) &= \text{In} \ - \ \text{Out} \ = \left(\frac{b}{N}\right) I(t) \times S(t) - \gamma \times I(t) \\
R'(t) &= \text{In} \ - \ \text{Out} \ = \gamma \times I(t) - 0
\end{aligned}
$$

SIR Model:

$$
\begin{cases}
S' = -\left(\frac{b}{N}\right) SI \\
I'(t) = \left(\frac{b}{N}\right) SI - \gamma I \\
R'(t) = \gamma I
\end{cases}
$$

Note: If you add up all three equations, you get

$$(S+I+R)' = -\left(\frac{b}{N}\right)SI + \left(\frac{b}{N}\right)SI - \gamma I + \gamma I = 0$$

So the total population $N = S + I + R$ is constant.

Simplification: Since $S + I + R = N$ we can just focus on S and I and calculate R from $R = N - S - I$

Reduced Model:

$$\begin{cases} S' = -\left(\frac{b}{N}\right)SI \\ I' = \left(\frac{b}{N}\right)SI - \gamma I \end{cases}$$

5. Re-scaling Time

To simplify this further, let's re-scale time

That is, define a new unit of time τ so that

$$\tau = 1 \Leftrightarrow t = \frac{1}{\gamma} \text{ (mean residence time of the disease) } \Leftrightarrow \gamma t = 1$$

Therefore define $\boxed{\tau = \gamma t}$

And define x and y to be $\frac{S}{N}$ and $\frac{I}{N}$ but with our new variables

$$x(\tau) = \frac{S(t)}{N} = \frac{S\left(\frac{\tau}{\gamma}\right)}{N}$$
$$y(\tau) = \frac{I(t)}{N} = \frac{I\left(\frac{\tau}{\gamma}\right)}{N}$$

Interpretation: x and y represent the fraction of people that are susceptible resp. infected

Rewrite SIR in terms of x and y

$$\textbf{Note: } x(\tau) = \frac{S(t)}{N} \Rightarrow S(t) = Nx(\tau) = Nx(\gamma t)$$

$$\text{Therefore } S'(t) = (Nx(\gamma t))' = Nx'(\gamma t)\,(\gamma t)' = \gamma N x'(\tau)$$

And our ODE becomes

$$\begin{aligned} S'(t) &= -\left(\frac{b}{N}\right) S(t)I(t) \\ \gamma \cancel{N} x'(\tau) &= -\left(\frac{b}{\cancel{N}}\right) (\cancel{N}x(\tau))\,(\cancel{N}y(\tau)) \\ \gamma x'(\tau) &= -\,bx(\tau)y(\tau) \end{aligned}$$

Similarly for $I'(t)$ and so the new re-scaled model becomes:

$$\begin{cases} x' = -\left(\frac{b}{\gamma}\right) xy \\ y' = \left(\frac{b}{\gamma}\right) xy - y \end{cases}$$

So if we let $\boxed{R_0 = b/\gamma}$ then we get

Re-Scaled Model:

$$\begin{cases} x' = -R_0 xy \\ y' = R_0 xy - y \end{cases}$$

Interpretation of R_0: Notice $R_0 = b \times (1/\gamma)$

b is the number of people a sick person infects per day

$1/\gamma$ is the duration of an infection.

Hence R_0 is the number of people that get infected while someone is infectious

Note: $R_0 > 1$ means each sick person infects at least one other person over the course of being sick. It will be interesting to distinguish the cases $R_0 < 1$ and $R_0 > 1$ and we will indeed do so later