LECTURE: EPIDEMIOLOGY: SIR MODELS

1. Recap: SIR Model

Last time: Used SIR model (Suspected, Infected, Recovered) to model the spread of COVID. After eliminating one variable and rescaling time, we got the following:

Our Model:

$$\begin{cases} x' = -R_0 xy \\ y' = R_0 xy - y \end{cases}$$

$x =$ fraction of suspected people

$y =$ fraction of infected people

$R_0 =$ number of people a sick person infects while they're sick

Interpretation of R_0: Notice $R_0 = b \times (1/\gamma)$

b is the number of people a sick person infects per day

$1/\gamma$ is the duration of an infection.

Hence R_0 is the number of people that get infected while someone is infectious

Note: $R_0 > 1$ means each sick person infects at least one other person over the course of being sick. It will be interesting to distinguish the cases $R_0 < 1$ and $R_0 > 1$ and we will indeed do so below

2. Equilibrium Points

Set $x' = 0$ and $y' = 0$ to get

$$\begin{cases} -R_0 xy = 0 \\ R_0 xy - y = 0 \end{cases} \Rightarrow \begin{cases} xy = 0 \\ y(R_0 x - 1) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \text{ or } y = 0 \\ y(R_0 x - 1) = 0 \end{cases}$$

Case 1: $x = 0$

Then the second equation becomes

$$y\left(R_0 0 - 1\right) = 0 \Rightarrow -y = 0 \Rightarrow y = 0 \Rightarrow (0,0)$$

Case 2: $y = 0$

Then the second equation becomes

$$0\left(R_0 x - 1\right) = 0 \Rightarrow 0 = 0$$

Which is true *no matter what* the value of x is $\Rightarrow (x, 0)$

Equilibrium Points:

$(x, 0)$ where x is arbitrary

Note: This includes $(0,0)$ if you let $x = 0$

In other words, the whole $x-$axis is made out of equilibrium points!!

Interpretation: $(x, 0)$ means no one is infected since $y = 0$. This happens either before or during the early stages of the pandemic.

3. Classification

$$\nabla F(x,y) = \begin{bmatrix} \frac{\partial(-R_0xy)}{\partial x} & \frac{\partial(-R_0xy)}{\partial y} \\ \frac{\partial(R_0xy-y)}{\partial x} & \frac{\partial(R_0xy-y)}{\partial y} \end{bmatrix} = \begin{bmatrix} -R_0y & -R_0x \\ R_0y & R_0x-1 \end{bmatrix}$$

$$\nabla F(x,0) = \begin{bmatrix} -R_0(0) & -R_0x \\ R_0(0) & R_0x-1 \end{bmatrix} = \begin{bmatrix} 0 & -R_0x \\ 0 & R_0x-1 \end{bmatrix}$$

The eigenvalues are $\lambda = 0$ (neutral) and $\lambda = R_0x - 1$

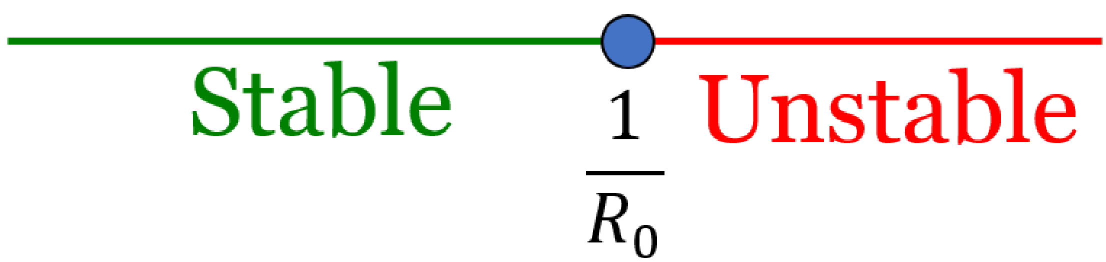
Note: $R_0x - 1 = 0 \Rightarrow R_0x = 1 \Rightarrow x = \frac{1}{R_0}$ In particular

$$\lambda = R_0x - 1 = \begin{cases} < 0 & \text{for } x < \frac{1}{R_0} \\ > 0 & \text{for } x > \frac{1}{R_0} \end{cases}$$

Conclusion:

$$(x,0) \text{ is } \begin{cases} \text{stable} & \text{if } x < 1/R_0 \\ \text{unstable} & \text{if } x > 1/R_0 \end{cases}$$

(If $x = 1/R_0$, then $(x,0)$ is neutral, neither stable nor unstable)

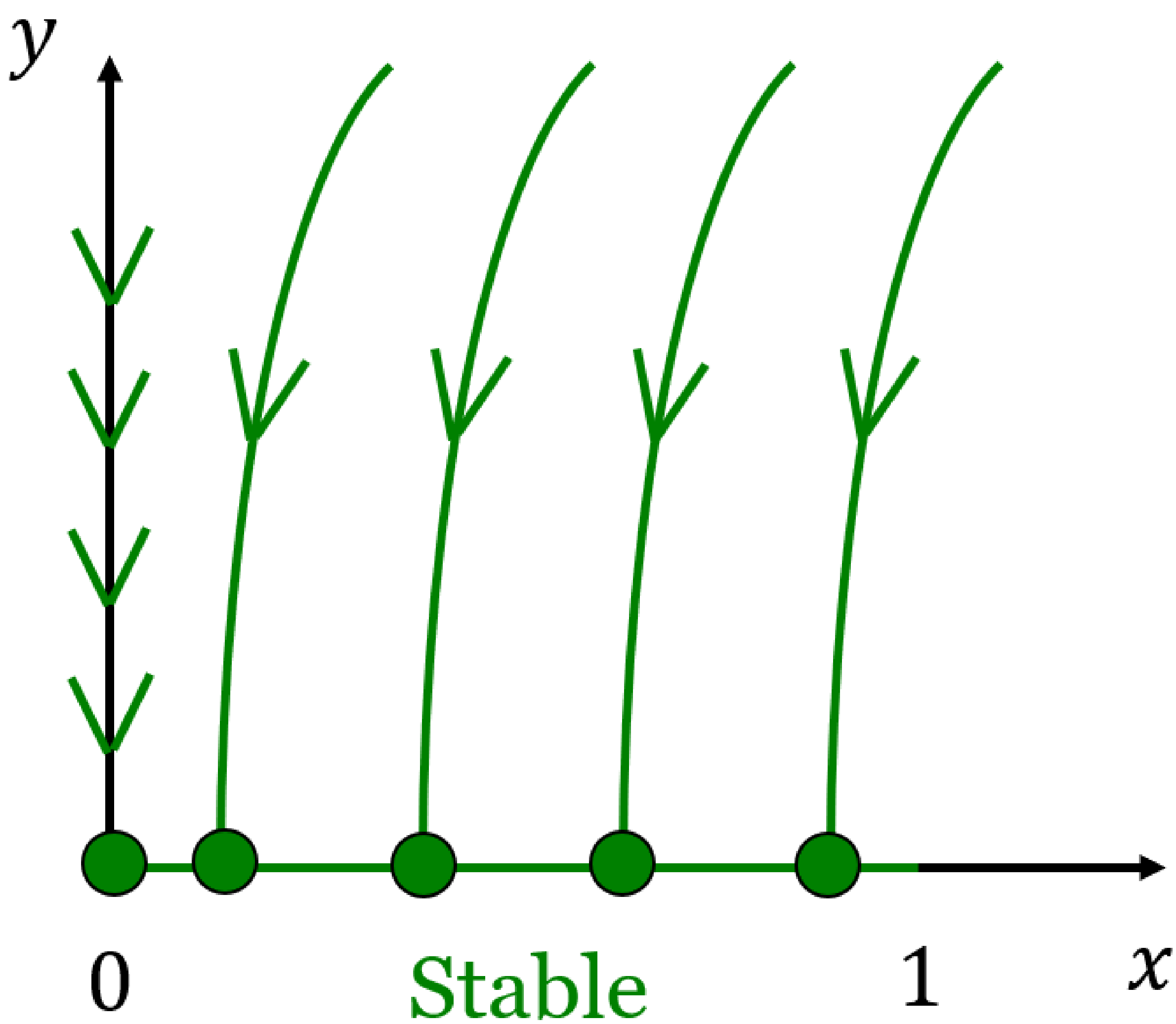


Here is where we'll distinguish the cases $R_0 < 1$ and $R_0 > 1$

4. The good case: $R_0 < 1$

R_0 is the number of people a sick person infects, so $R_0 < 1$ means an infected individual infects at most one person while being sick.

Note: By definition, x and y are between 0 and 1



$R_0 < 1 \Rightarrow \frac{1}{R_0} > 1$ so, since $x \leq 1$, all our equilibria $(x, 0)$ are stable!

Interpretation: This is the good case: The solutions eventually go to $(x, 0)$, so eventually, no one is infected any more ($y = 0$) even though some suspected cases may remain. In particular, there are no outbreaks = sudden rise in infections.

This makes sense if you remember what R_0 is. If a sick person infects less than 1 person on average, the disease doesn't really spread.

Justifications for phase portrait:

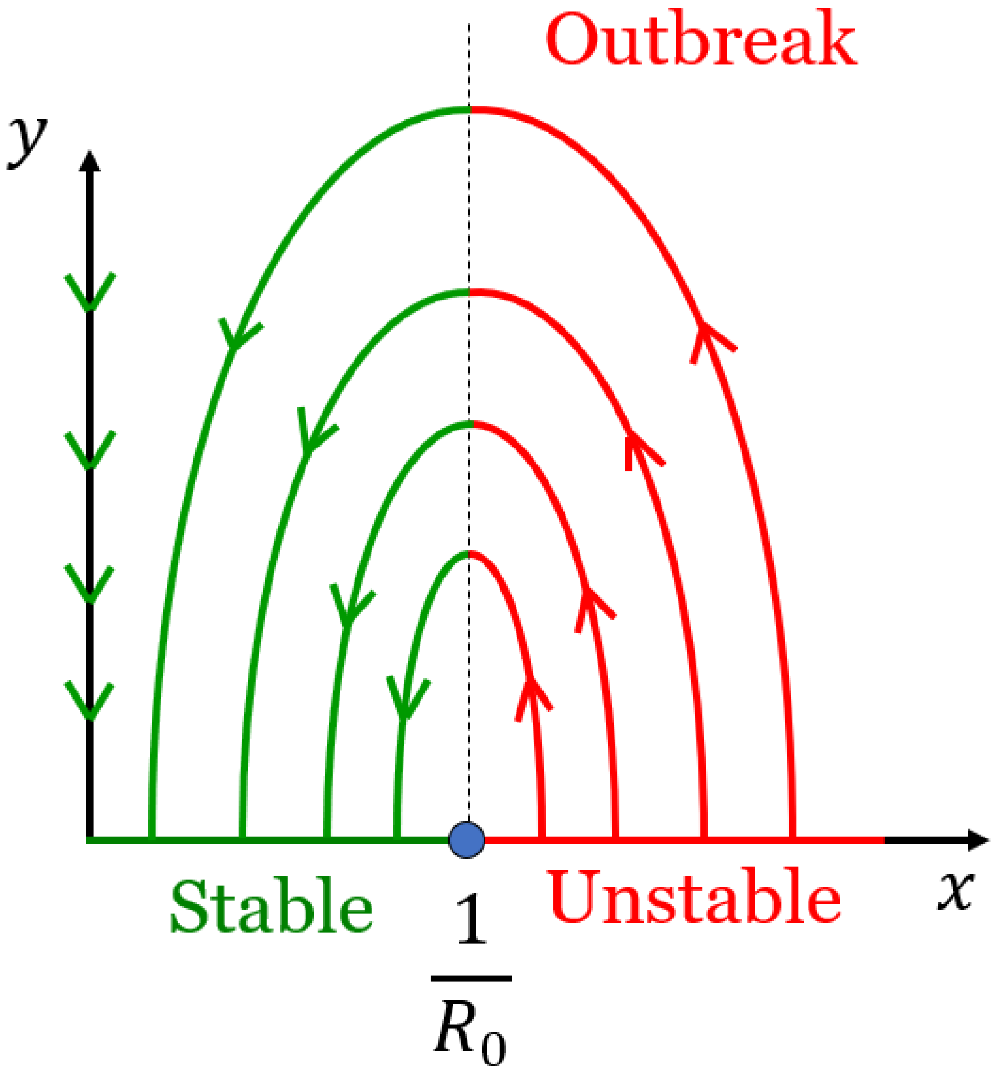
To draw this, we used $x' = -R_0 xy \leq 0$ (arrows point to the left) and

$$y' = y\left(R_0 \underbrace{x}_{\leq 1} - 1\right) \leq \underbrace{y}_{\geq 0}\left(\underbrace{R_0 - 1}_{\leq 0}\right) \leq 0 \text{ (arrows point down)}$$

On the $y-$axis, the arrows are pointing down because if $x = 0$ then $y' = 0 - y = -y < 0$

5. The bad case: $R_0 > 1$

This is where all hell breaks loose! On the right of $\frac{1}{R_0}$, all equilibrium points are unstable (solutions move out of them) while on the left they are stable (solutions move towards them), and we get **outbreaks**



Justifications for phase portrait:

Once again we have $x' \leq 0$ and

$$y' = \underbrace{y}_{\geq 0}(R_0 x - 1) = \begin{cases} > 0 \text{ if } x > \frac{1}{R_0} \\ < 0 \text{ if } x < \frac{1}{R_0} \end{cases}$$

Interpretation: If $R_0 > 1$, we may have **outbreaks**, where the number of infected people suddenly increases, even though it will eventually decrease, at least according to our model.

This once again makes sense if you remember what R_0 is. If, on average, a sick person infects more than 1 person, then the disease will spread throughout the population, leading to an outbreak. However, this does not happen if $x < \frac{1}{R_0}$, that is, when the number of suspected people is relatively small. Think for example that they are quarantined, or the disease is under control.

6. Effect of Vaccines

There is a very simple condition that guarantees that $x < \frac{1}{R_0}$

Definition:

$$z = \frac{R}{N} = \text{ fraction of recovered/vaccinated people}$$

Notice in particular that by definition, $x + y + z = 1$

Fact:

If $z > 1 - \frac{1}{R_0}$ then $x < \frac{1}{R_0}$

So if enough people are vaccinated, then we don't have an outbreak since we get herd immunity.

Why? Notice $z > 1 - \frac{1}{R_0} \Rightarrow 1 - z < \frac{1}{R_0}$ and so

$$x = 1 - \underbrace{y}_{\geq 0} - z \leq 1 - z < \frac{1}{R_0} \checkmark$$

Comparison with real data:

According to the New York Times data from Spring 2020, we have $R_0 \approx 3.8$ so in this case $1 - \frac{1}{R_0} = 1 - \frac{1}{3.8} \approx 0.74$, therefore at least 74% of the population needs to be vaccinated/immune from COVID-19 in order to get herd immunity. In comparison, only 69% of people in the US are fully vaccinated, so we're not *quite* there yet.

The End