APMA 1650 – HOMEWORK 10

Problem 1: Once again, you are the quality control manager for the Acme Widget Company. Your line of MiniWidgets has been so successful that the MiniWidget machines are running nonstop to satisfy the high customer demand. For a properly functioning MiniWidget machine, the probability of producing a defective MiniWidget is 1% (or less). As part of the quality control process, you will take a sample of 100 MiniWidgets from a machine to determine whether it needs repair. To make a statistically-sound decision, you design a hypothesis test to aid you in this process. You desire a level of $\alpha = 0.05$ for your hypothesis test.

(a) State the alternative hypothesis, null hypothesis, and test statistic for your hypothesis test.

We are interested in the proportion of defective widgets produced by the machine. Alternative hypothesis: $p > 0.01$. Null hypothesis: $p = 0.01$. Test statistic $\hat{p} = Y/n$, where $n = 100$ and Y is the number of defective widgets in your sample of 100.

(b) You sample 100 MiniWidgets from one of your machines and find that 3 of them are defective. At the level of $\alpha = 0.05$, does this machine need to be repaired?

To compute the rejection region, we need to find the standard deviation of our test statistic $\hat{p}$. Since we don't know the true population proportion p, we will estimate it with the sample proportion $\hat{p}$ since the sample size is large. In this case $\hat{p} =$

$3/100 = 0.03$

$$\begin{aligned}\sigma_{\hat{p}} &= \sqrt{\frac{p(1-p)}{n}}\\ &\approx \sqrt{\frac{\hat{p}(1-h\hat{p})}{n}}\\ &= \sqrt{\frac{(0.03)(0.97)}{100}}\\ &= 0.017\end{aligned}$$

The rejection region is given by:

$$\begin{aligned}\hat{p} &= p_0 + z_\alpha \sigma_{\hat{p}}\\ &= 0.01 + 1.65(0.017)\\ &= 0.038\end{aligned}$$

Since $\hat{p}$ does not fall inside the rejection region, we do not reject the null hypothesis, thus we conclude with level $\alpha = 0.05$ that the machine does not need to be repaired.

(c) What is the p-value for this hypothesis test?

The p-value is the smallest value of α for which we will reject the null hypothesis.

$$\begin{aligned}p - \text{value} &= \mathbb{P}(\hat{p} \geq 0.03 \text{ given } p = p_0 = 0.01)\\ &= \mathbb{P}\left(\frac{\hat{p}-p_0}{\sigma_{\hat{p}}} \geq \frac{0.03-p_0}{\sigma_{\hat{p}}}\right)\\ &= \mathbb{P}\left(Z \geq \frac{0.03-0.01}{0.017}\right)\\ &= \mathbb{P}(Z \geq 1.18)\\ &= 0.1190\end{aligned}$$

In this case, since we do not reject the null hypothesis at a level $\alpha = 0.05$, the p-values is greater than this value of α.

Problem 2: A random sample of 500 measurements on the length of stay in hospitals had a sample mean of 5.4 days and a sample standard deviation of 3.1 days. A federal regulatory agency hypothesizes that the average length of stay is greater than 5 days.

(a) Do the data support this hypothesis with a level of $\alpha = 0.05$?

The null hypothesis is $\mu = 5$, alternative hypothesis is $\mu > 5$, and test statistic is $\bar{Y}$. For the rejection region, we will approximate the population standard deviation σ with the sample standard deviation S since the sample size is large. The rejection region is given by:

$$\begin{aligned}\bar{Y} &\geq \mu_0 + z_\alpha \frac{S}{\sqrt{n}} \\ &= 5 + 1.65 \frac{3.1}{\sqrt{500}} \\ &= 5.23\end{aligned}$$

Since the rejection region is $\bar{Y} \geq 5.23$, our value of $\bar{Y}$ falls within the rejection region, thus we reject the null hypothesis, and so the claim is supported by the data at a level of $\alpha = 0.05$.

(b) What is the p-value for this hypothesis test?

For the p-value:

$$\begin{aligned} p-\text{value} &= \mathbb{P}(\bar{Y} \geq 5.4 \text{ given } \mu = \mu_0 = 5) \\ &= \mathbb{P}\left(\frac{\bar{Y}-\mu_0}{S/\sqrt{n}} \geq \frac{5.4-\mu_0}{S/\sqrt{n}}\right) \\ &= \mathbb{P}\left(Z \geq \frac{5.4-5.0}{3.1/\sqrt{500}}\right) \\ &= \mathbb{P}(Z \geq 2.89) \\ &= 0.0019 \end{aligned}$$

(c) Using the rejection region found in the previous part, calculate β for the specific value of the alternative hypothesis $\mu_a = 5.5$.

$$\begin{aligned} \beta &= \mathbb{P}\left(Z \leq \frac{k-\mu_a}{S/\sqrt{n}}\right) \\ &= \mathbb{P}\left(Z \leq \frac{5.23-5.5}{3.1/\sqrt{500}}\right) \\ &= \mathbb{P}(Z \leq -0.41) \\ &= \mathbb{P}(Z \leq -1.95) \\ &= 0.0256 \end{aligned}$$

(d) How large should the sample size be if we require that $\alpha = 0.01$ and $\beta = 0.05$, where we use the specific value of the alternative hypothesis $\mu_a = 5.5$.

Using the formula from the notes, approximating σ^2 with the sample variance S^2, we get

$$\begin{aligned} n &= \frac{(z_\alpha + z_\beta)^2 S^2}{(\mu_a - \mu_0)^2} \\ &= \frac{(2.33 + 1.65)^2 3.1^2}{(5.5 - 5.0)^2} \\ &= 608.905 \end{aligned}$$

Rounding up, we should have a sample size of 609.

Problem 3: You are naturalist studying feeding habits of white-tailed deer. You have noticed that these deer live and feed within relatively narrow ranges, approximately 150 to 200 acres (there are 640 acres per square mile, so these ranges are indeed small!) You study two geographically isolated populations of white-tailed deer and measure the distance they range by using small, radio transmitters that you attach to each deer. (No deer are harmed in the course of your study.) For each of the two populations, you study 40 deer. To quantify the ranges for each deer, you measure the distance Y between where the deer was released after being fit with the radio transmitter and where the radio transmitter was found one month later. The following table gives the data from the study:

	Location 1	Location 2
Sample size	40	40
Sample mean (feet)	2980	3205
Sample standard deviation (feet)	1140	963

You wish to determine statistically whether there is any difference in the ranges of the two deer populations.

(a) What is the parameter of interest in this study?

You are interested in the difference between the population means $\mu_1 - \mu_2$.

(b) What is the null hypothesis, alternative hypothesis, and test statistic?

The null hypothesis is $\mu_1 - \mu_2 = 0$. The alternative hypothesis is $\mu_1 - \mu_2 \neq 0$. The test statistic is $\bar{Y}_1 - \bar{Y}_2$.

(c) Do the data provide sufficient evidence that the mean ranges of the two populations are different? Use a level of $\alpha = 0.10$ for your hypothesis test.

For this, we do a two-tailed hypothesis test, since we have no reason to suspect that one population has a larger range than the other. First, we need the standard deviation of the estimator $\bar{Y}_1 - \bar{Y}_2$. Using the formula from the table in the notes, and estimating the population standard deviations with the sample standard deviations:

$$\sigma_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \approx \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$
$$= \sqrt{\frac{1140^2}{40} + \frac{963^2}{40}} = 236$$

Since this is a two-tailed test, we split our α. From the Z-table, $z_{\alpha/2} = z_{0.05} = 1.64$. (You could also use 1.64). Since the null hypothesis is that the difference in means is 0, the rejection region is:

$$|(\bar{Y}_1 - \bar{Y}_2) - 0| \geq \sigma_{\bar{Y}_1 - \bar{Y}_2} z_{\alpha/2}$$
$$|\bar{Y}_1 - \bar{Y}_2| \geq 236(1.64) = 387$$

For our samples, $|\bar{Y}_1 - \bar{Y}_2| = |2980 - 3205| = 225$. This does not fall in the rejection region, so at the level of 0.10, we fail to reject the null hypothesis, so there is not sufficient evidence that the mean ranges are different.

Problem 4: You are for one final time the quality control manager for the Acme Widget Company. Since MiniWidgets were so successful, you decide to launch a line of MegaWidgets. Same great idea, (approximately) 100 times the size! Your MegaWidget machine is designed to produce MegaWidgets which have an average mass of 800 grams. You suspect that your MegaWidget machine is producing MegaWidgets which are too small. Assume that the masses of the MegaWidgets produced by the machine are normally distributed. Since MegaWidgets are more expensive and take longer to produce than MiniWidgets, you decide to take a sample of 5 MegaWidgets from the machine. Their masses (in grams) are 785, 805, 790, 793, and 802 grams.

Do the data indicate that the average mass of MegaWidgets produced by the machine is less than 800 grams? Use a hypothesis test with a level of significant $\alpha = 0.05$.

The sample mean is $\bar{Y} = 795$, and the sample standard deviation (using the unbiased estimator) is $S = 8.337$. For the hypothesis test, the null hypothesis is $\mu = 800$, the alternative hypothesis is $\mu < 800$, and the test statistic is $\bar{Y}$. This is a lower-tail test, and since we have a normally distributed population (we need to assume this!), unknown population variance, and a small sample, we will use the t-test. We have 5 samples, so we need to use the t-distribution with $5 - 1 = 4$ df. Looking at the t-table (and noting that the distribution is symmetric about the mean), we have $t_\alpha = t_{0.05} = 2.132$. The rejection region is

$\bar{Y} \leq k$, where k is given by:

$$k = \mu - t_\alpha \frac{S}{\sqrt{n}}$$
$$= 800 - 2.133 \frac{8.337}{\sqrt{5}} = 792$$

note that we use the value of $\mu = 800$ from the null hypothesis. Since $795 > 792$, the test statistic does not fall in the rejection region, so we fail to reject the null hypothesis with a level of 0.05. Thus we conclude that we do not need to repair the MegaWidget machine at present.