LECTURE: HYPOTHESIS TESTING (II)

1. Large Sample Hypothesis Tests

Goal: Hypothesis Testing when the sample is large

Setting: Parameter of interest θ

(1) Alternative hypothesis: $H_a : \theta > \theta_0$

(2) Null hypothesis, $H_0 : \theta = \theta_0$

(3) Test statistic, $\hat{\theta}$ some *unbiased* estimator

(4) Rejection region (RR) $\hat{\theta} \geq k$ where k is TBA

Problem: How to find k?

Recall:

$$\alpha = P(\text{ reject Null when Null is true })$$

Goal: We're given α (think tolerance) and want to find k

STEP 1: If the sample is large (CLT) and null $\theta = \theta_0$ is true then

$$\hat{\theta} \sim N\left(\theta_0, \hat{\sigma}^2\right)$$

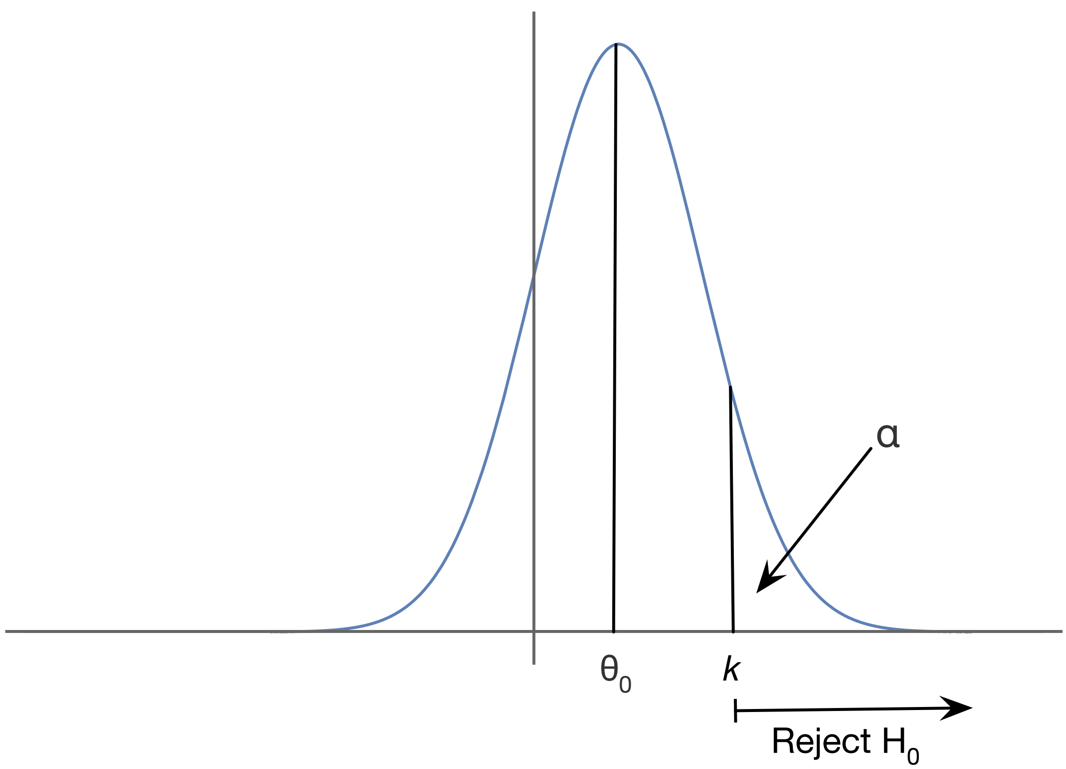
Here $\hat{\sigma}$ is the standard deviation of $\hat{\theta}$ and we used $E(\hat{\theta}) = \theta = \theta_0$

STEP 2: By def of α and converting to Z we want to choose k with

$$P(\hat{\theta} \geq k) = \alpha \Rightarrow P\left(\frac{\hat{\theta} - \theta_0}{\hat{\sigma}} \geq \frac{k - \theta_0}{\hat{\sigma}}\right) = \alpha \Rightarrow P\left(Z \geq \frac{k - \theta_0}{\hat{\sigma}}\right) = \alpha$$

STEP 3: Using the Z table, we find z_α with $P(Z \geq z_\alpha) = \alpha$ hence

$$\frac{k - \theta_0}{\hat{\sigma}} = z_\alpha \Rightarrow k = \theta_0 + z_\alpha \, \hat{\sigma}$$



Example 1:

Ice cream preferences in the US

$p =$ the proportion of people who prefer chocolate ice cream.

Suppose you sample 100 people and 60 of them favor chocolate

Does the evidence support chocolate being favored at $\alpha = 0.05$?

Alternative hypothesis: $p \geq 0.5$

Null Hypothesis: $p = 0.5$

Test Statistic: $\hat{p} = \frac{60}{100} = 0.6$

RR: $\alpha = 0.05$ and by the $Z-$table $P(Z \geq 1.64) \approx 0.05$ so $z_\alpha = 1.64$

$$\hat{\sigma} = \text{ Standard Dev } (\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{(0.5)(0.5)}{100}} = 0.05$$

(We used $\hat{p} = \frac{Y}{n}$ and $Y \sim$ Binom (n, p) and $p = 0.5$ by Null Hyp)

$$k = 0.5 + z_\alpha \, \hat{\sigma} = 0.5 + 1.64(0.05) = 0.582$$

Thus the rejection region is $\{\hat{p} \geq 0.582\}$

Answer: $\hat{p} = 0.6$ falls inside the rejection region, thus we can reject our null hypothesis with a level of 0.05, we are 95% confident that chocolate is favored

2. Two-Tailed Hypothesis Tests

Similarly, we can do this for a two-tailed hypothesis test $\left|\hat{\theta} - \theta_0\right| \geq k$

STEP 1:

$$P(|\hat{\theta} - \theta_0| \geq k) = \alpha \Rightarrow P\left(\left|\frac{\hat{\theta} - \theta_0}{\hat{\sigma}}\right| \geq \frac{k}{\hat{\sigma}}\right) = \alpha \Rightarrow P\left(|Z| \geq \frac{k}{\hat{\sigma}}\right) = \alpha$$

STEP 2: By the method of splitting the difference (see picture) it is enough to have

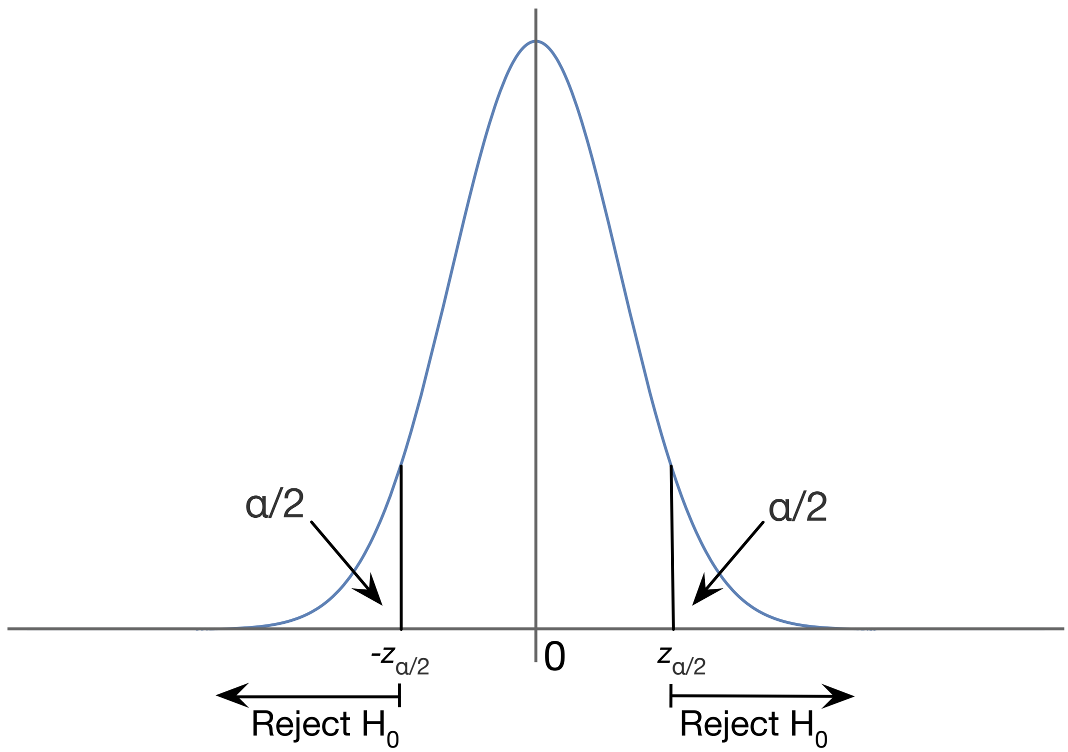
$$P\left(Z \leq -\frac{k}{\hat{\sigma}}\right) = \alpha/2 \text{ and } P\left(Z \geq \frac{k}{\hat{\sigma}}\right) = \alpha/2$$

STEP 3: Use the Z table to find $z_{\alpha/2}$ and therefore

$$\frac{k}{\hat{\sigma}} = z_{\alpha/2} \Rightarrow k = z_{\alpha/2}\,\hat{\sigma}$$

STEP 4: Hence our rejection region is

$$\{|\hat{\theta} - \theta_0| \geq z_{\alpha/2}\hat{\sigma}\}$$



Example 2:

You're designing a ball bearing machine which produces ball bearings that are 5 mm in diameter, but you suspect that there is something wrong with the machine

You sample 64 ball bearings and obtain a sample mean of 4.98 mm and a sample standard deviation of 0.1 mm

Can you conclude at a level of 0.95 that there is something wrong with the machine?

Alternative hypothesis: $\mu \neq 5$

Null Hypothesis: $\mu = 5$

Test Statistic: $\bar{Y} = 4.98$

RR: $\alpha = 0.05 \Rightarrow \alpha/2 = 0.025$ and $P(Z \geq 1.96) \approx 0.025$ so $z_{\alpha/2} = 1.96$

Since do not know σ we use S in place of σ

$$\hat{\sigma} \approx \frac{S}{\sqrt{n}} = \frac{0.1}{\sqrt{64}} = \frac{0.1}{8} = 0.0125$$

$$k = z_{\alpha/2}\,\hat{\sigma} = (1.96)(0.0125) = 0.0245$$

The rejection region is therefore:

$$\{|\bar{Y} - 5| \geq 0.0245\} = \{\bar{Y} \leq 4.9755 \text{ or } \bar{Y} \geq 5.0245\}$$

Answer: Since $\overline{Y} = 4.98$ is not in our rejection region, we do not reject the null hypothesis; you do not have to do maintenance on the machine.

3. Type II Error

So far: We have fixed α and found k such that RR is $\left\{\hat{\theta} \geq k\right\}$

Goal: Find the type II error β

Recall:

$$\begin{aligned}\beta =& P(\text{ accept Null when Null is false })\\ =& P(\hat{\theta} \text{ lies outside RR when Alt is true })\end{aligned}$$

Same setting as before, with $RR \left\{ \hat{\theta} \geq k \right\}$

Definition:

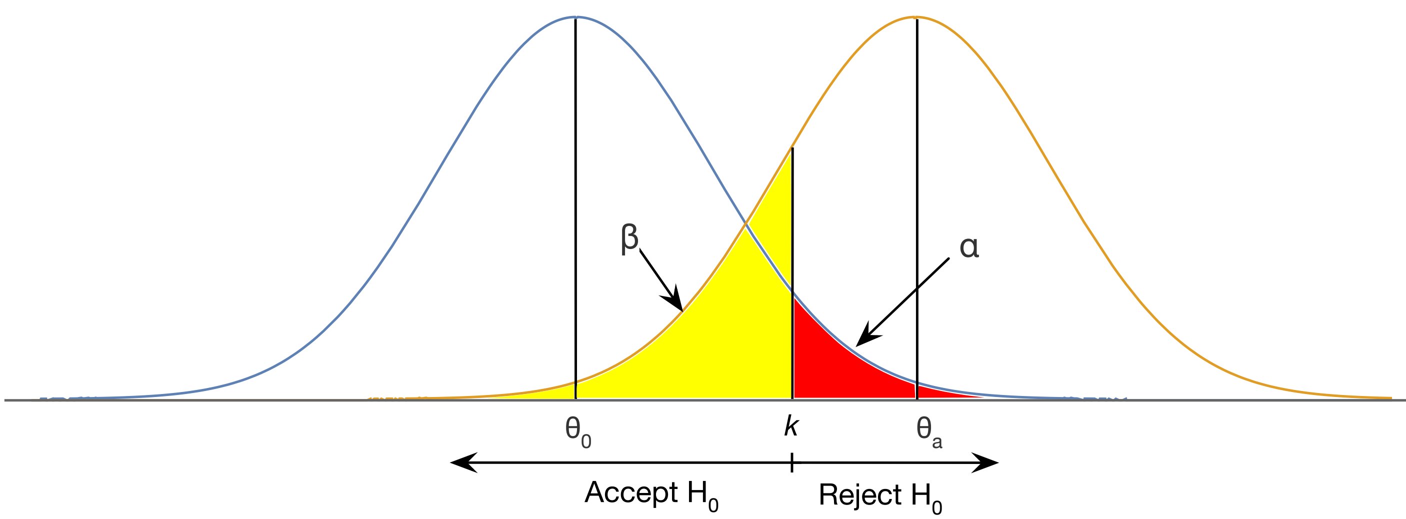
θ_a = the true (alternative) value of θ

Example: In the ice cream example we had $\theta_0 = 0.5$ but the true value may have been $\theta_a = 0.7$

By definition of β we have $\beta = P(\hat{\theta} \leq k \text{ given } \theta = \theta_0)$

Upshot: Now we have $\theta = \theta_a$ so $\hat{\theta} \sim N(\theta_a, \hat{\sigma}^2)$ and converting to Z

$$\beta = P(\hat{\theta} \leq k) = P\left(\frac{\hat{\theta} - \theta_a}{\hat{\sigma}} \leq \frac{k - \theta_a}{\hat{\sigma}}\right) = P\left(Z \leq \frac{k - \theta_a}{\hat{\sigma}}\right)$$



Example 3:

You're polling the cream preferences in the US.

$p =$ the proportion of chocolate supporters

You sample 100 people and 60 of them prefer chocolate

As before, you create a hypothesis test with $\alpha = 0.05$, for which the RR is $\{\hat{p} \geq 0.582\}$

Calculate β given $p_a = 0.60$

In this case, since $p_a = 0.60$ we estimate $\hat{\sigma}$ using $p = p_a$

$$\hat{\sigma} \approx \sqrt{\frac{p_a(1-p_a)}{n}} = \sqrt{\frac{(0.6)(0.4)}{100}} = 0.049$$

$$\beta = P\left(Z \leq \frac{k - p_a}{\hat{\sigma}}\right) = P\left(Z \leq \frac{0.58 - 0.60}{0.049}\right) = P(Z \leq -0.41) = 0.3409$$