

LECTURE: LIKELIHOOD RATIO TESTS

1. INTRODUCTION

Recall:

The **power of a hypothesis test** is

$$\text{Power}(\theta) = P(\hat{\theta} \text{ lies in RR when true parameter value is } \theta)$$

Recall:

The **most powerful α -level test** is a hypothesis test such that

- (1) $\text{Power}(\theta_0) = \alpha$
- (2) $\text{Power}(\theta_a)$ is as large as possible

So far: We constructed the most powerful α -level test in the case of a simple hypothesis (= hyp determines population distribution)

Goal: What happens if we have a composite hypothesis?

Example: Normal distribution, where there are **two** unknown parameters, the mean μ **and** the variance σ^2 so a hypothesis test of the form “ $\mu = 5$ ” is composite.

In this case, we use a **likelihood ratio test**

2. SET-UP

STEP 1: Given a population which is parameterized by a set of parameters $\Theta = (\theta_1, \dots, \theta_n)$

Example: The normal distribution is parameterized by $\Theta = (\mu, \sigma^2)$

STEP 2: Take iid samples Y_1, \dots, Y_n

STEP 3: Given values of the parameters $\Theta = (\theta_1, \dots, \theta_n)$ the likelihood function for our sample is $L(Y_1, \dots, Y_n | \Theta)$

STEP 4:

Alt states that Θ lies in a particular set Ω_a of values

Null states that Θ lies in another set Ω_0

By definition of Alt and Null Ω_0 and Ω_a must be disjoint

Example 1:

Suppose our population $\sim \text{Exp}(\lambda)$ and our hypotheses are:

Alt: $H_a : \lambda \neq 5$

Null: $H_0 : \lambda = 5$

Then $\Omega_a = \{\lambda > 0 : \lambda \neq 5\}$ and $\Omega_0 = \{5\}$

STEP 5: Parameter Space

Definition:

The **parameter space** is $\Omega = \Omega_0 \cup \Omega_a$

This is all possible values of the parameters

Example: In the exponential example above $\Omega = \{\lambda > 0\}$

STEP 6: Likelihoods**Definition:**

$$L(\hat{\Omega}_0) = \max_{\Theta} L(Y_1, \dots, Y_n | \Theta) \text{ with } \Theta \in \Omega_0$$

$$L(\hat{\Omega}) = \max_{\Theta} L(Y_1, \dots, Y_n | \Theta) \text{ with } \Theta \in \Omega$$

Here $L(\hat{\Omega}_0)$ as the “best explanation” for the observed data given Null is true and $L(\hat{\Omega})$ is the “best explanation” for the observed data given all possible values of Θ .

So if $L(\hat{\Omega}_0) = L(\hat{\Omega})$ then the best explanation is the null hypothesis, so we should accept null

And if $L(\hat{\Omega}_0) < L(\hat{\Omega})$ the best explanation is found inside Ω_a , so we should reject null in favor of the alternative hypothesis.

STEP 7: Likelihood ratio test

Definition:

Given the set-up above, the **likelihood ratio** λ is

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}$$

Definition:

A **likelihood ratio test** is a hyp test with the following set-up

- (1) Alternative hypothesis: $H_a : \Theta \in \Omega_a$
- (2) Null Hypothesis: $H_0 : \Theta \in \Omega_0$
- (3) Test Statistic: λ
- (4) Rejection region $\{\lambda \geq k\}$

Here k is chosen so that α is a desired level.

Interpretation: It can be shown that $0 \leq \lambda \leq 1$. A value of λ close to 0 indicates that the likelihood of the sample is much smaller under H_0 than H_a , which favors rejection of the null hypothesis.

Example 2:

Let Y_1, \dots, Y_n be iid samples from $N(\mu, \sigma^2)$ where both μ and σ^2 are unknown.

Suppose we wish to test Alt $H_a : \mu > \mu_0$ with Null $H_0 : \mu = \mu_0$

Then if we use the likelihood ratio test described above, we obtain a hypothesis test with test statistic λ and rejection region $\lambda > k$ with

$$k = \mu_0 + t_\alpha \left(\frac{S}{\sqrt{n}} \right)$$

Thus the likelihood ratio test for this scenario is exactly the t -test we discussed earlier. The proof of this is messy and will be omitted

Unfortunately, in most cases, the likelihood ratio does not give us a known distribution such as the t -distribution.

3. CHI-SQUARE DISTRIBUTION

It can be shown, however, if the sample size is large and the underlying distribution is “nice” the likelihood ratio λ has a chi-square distr:

Fact:

Let r_0 be the number of parameters in Null $H_0 : \Theta \in \Omega_0$

Let r be the number of parameters in $\Omega = \Omega_0 \cup \Omega_a$

Then, if n is large, the test statistic

$$-2 \ln(\lambda)$$

Has an approximately chi-square distribution with $r - r_0$ df

Consequence: In this case, the rejection region is given by

$$-2 \ln(\lambda) > \chi_\alpha^2$$

Where χ_α^2 is found in the chi-square table based on $r - r_0$ df

Example 3:

You are the quality control engineer for the ACME widget factory, and you want to compare the number of defective widgets produced per day by two different factories.

You observe the factories for 100 days and find sample means $\bar{X} = 20$ and $\bar{Y} = 22$

Assume the number of defective widgets produced per day by the first factory $\sim \text{Poi}(\theta_1)$ and for the second factory $\sim \text{Poi}(\theta_2)$

Use the likelihood ratio test to test the following with $\alpha = 0.01$

$$H_0 : \theta_1 = \theta_2 \text{ vs } H_a : \theta_1 \neq \theta_2$$

STEP 1:

Let X_1, \dots, X_n be the number of defective widgets from days 1 to n from the first factory, and Y_1, \dots, Y_n for the second factory

In this problem, $n = 100$

The population parameter is $\Theta = (\theta_1, \theta_2)$

The null hypothesis is $\Omega_0 = \{(\theta_1, \theta_2) : \theta_1 = \theta_2 = \theta\}$, where θ is unknown

STEP 2: Likelihood function

$$\begin{aligned}
L(\theta_1, \theta_2) &= L(X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n | \theta_1, \theta_2) \\
&= \left(\prod_{i=1}^{100} \frac{e^{-\theta_1} \theta_1^{X_i}}{X_i!} \right) \times \left(\prod_{i=1}^{100} \frac{e^{-\theta_2} \theta_2^{Y_i}}{X_i!} \right) \\
&= \left(\frac{1}{k} \right) \theta_1^{n\bar{X}} \theta_2^{n\bar{Y}} e^{-n(\theta_1 + \theta_2)}
\end{aligned}$$

Here $k = X_1! \cdots X_n! Y_1! \cdots Y_n!$ and $n = 100$

STEP 3: $L(\hat{\Omega}_0)$

If Null is true then $\theta_1 = \theta_2 = \theta$ and so

$$L(\theta) =: L(\theta, \theta) = \left(\frac{1}{k} \right) \theta^{n(\bar{X} + \bar{Y})} e^{-2n\theta}$$

In that case, you can show that the MLE is

$$\hat{\theta} = \frac{1}{2}(\bar{X} + \bar{Y})$$

Therefore, in this case we have

$$L(\hat{\Omega}_0) = \left(\frac{1}{k} \right) \left(\hat{\theta} \right)^{n(\bar{X} + \bar{Y})} e^{-2n\hat{\theta}} \text{ where } \hat{\theta} = \frac{1}{2}(\bar{X} + \bar{Y})$$

STEP 4: $L(\hat{\Omega})$

Our alternative hypothesis is $\Omega_a = (\theta_1, \theta_2) : \theta_1 \neq \theta_2$ therefore

$$\Omega = \Omega_0 \cup \Omega_a = \{(\theta_1, \theta_2) : \theta_1, \theta_2 > 0\}$$

$$L(\theta_1, \theta_2) = \left(\frac{1}{k} \right) (\theta_1)^{n\bar{X}} (\theta_2)^{n\bar{Y}} e^{-n(\theta_1 + \theta_2)}$$

Using multivariable calculus, one can show that $L(\theta_1, \theta_2)$ is maximized when both $\theta_1 = \bar{X}$ and $\theta_2 = \bar{Y}$

Therefore in this case, we get

$$L(\hat{\Omega}) = \left(\frac{1}{k}\right) (\bar{X})^{n\bar{X}} (\bar{Y})^{n\bar{Y}} e^{-n(\bar{X}+\bar{Y})}$$

STEP 5: Likelihood ratio

$$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} = \frac{\text{STEP 3}}{\text{STEP 4}} = \frac{(\hat{\theta})^{n(\bar{X}+\bar{Y})}}{\bar{X}^{n\bar{X}} + \bar{Y}^{n\bar{Y}}}$$

Here $n = 100$, $\bar{X} = 20$, $\bar{Y} = 22$, and $\hat{\theta} = \frac{1}{2}(\bar{X} + \bar{Y}) = 21$ so

$$\lambda = \frac{21^{100(20+22)}}{20^{(100)(20)} + 22^{(100)(22)}}$$

STEP 6: Chi-Square

$$-2 \ln(\lambda) = -2[4200 \ln(21) - 2000 \ln(20) - 2200 \ln(22)] = 9.53$$

By the Fact above, the above quantity has a $\chi_{0.01}^2$ distribution

The number of free parameters in $\Omega = \{(\theta_1, \theta_2) : \theta_1, \theta_2 > 0\}$ is 2

The number of free parameters in $\Omega_0 = \{(\theta_1, \theta_2) : \theta_1 = \theta_2 = \theta\}$ is 1

Thus $-2 \ln(\lambda)$ has a $\chi_{0.01}^2$ distribution with $2 - 1 = 1$ df

By the chi-square table, we have $\chi_{0.01}^2 = 6.635$ for 1 df

STEP 7: Answer

Hence the rejection region is

$$-2 \ln(\lambda) > 6.635$$

Since our value of $-2 \ln(\lambda) = 9.53$ lies inside the rejection region, we reject the null hypothesis at a significance level of 0.01

Answer: Thus we conclude that the number of defective widgets produced by the two factories is indeed different

The End