LECTURE: WHAT IS A ODE?

1. What is a differential equation?

Definition:

A differential equation (ODE) is an equation that involves one or more derivatives of a function y

Examples:

(1) Basic ODE: $y' = 2y$

(2) Second-order ODE: $y'' + 5y' + 6y = 0$

(3) Weird ODE: $(y''')^2 = \sin(y^3) + y + t^2$

(4) Systems of ODE:

$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = 5x(t) + 6y(t) \end{cases}$$

Here $x(t)$ and $y(t)$ are two functions that are coupled together

(5) PDE: $\frac{d^2u}{dxdt} + u = \left(\frac{du}{dx}\right)^2$

Here $u = u(x,t)$ studied in APMA 0360

Applications:

The real reason ODE are so powerful is because of their applications.

They are used to model and describe processes in:

(1) Biology (epidemiology, cancer research, COVID)
(2) Climate Research (wildfires)
(3) Economics (stock market, wealth inequality)
(4) Engineering and Physics (mass-spring systems, aircraft design)
(5) Neuroscience and Computer Science (deep learning, neural networks, traffic control)
(6) Modeling outbreak of a Zombie Attack
(7) Chemical Reactions: The PDE that got me the PhD

Basically, wherever something changes, there is a differential equation.

Finally, I would like to conclude with a word from my PhD advisor: "If you can solve all differential equations, then you can solve the universe." That is, not only are differential equations hard to solve, but also each equation is its own little universe.

2. The most basic example

Let's start this course by solving the most basic, yet most fundamental differential equation

Example 1:

$$\begin{cases} y' &= 2y \\ y(0) &= 3 \end{cases}$$

Application: This models for example the growth of bacteria: The more bacteria there are, the faster they reproduce.

STEP 1: Put everything on the left hand side

$$y' = 2y \Rightarrow \frac{y'}{y} = 2$$

STEP 2: Recognize the left-hand-side as a derivative

$$\begin{aligned} (\ln|y|)' &= 2 \\ \ln|y| &= 2t + C \\ |y| &= e^{2t+C} = e^C e^{2t} \\ y &= \underbrace{\pm e^C}_{C} e^{2t} \\ y(t) &= Ce^{2t} \end{aligned}$$

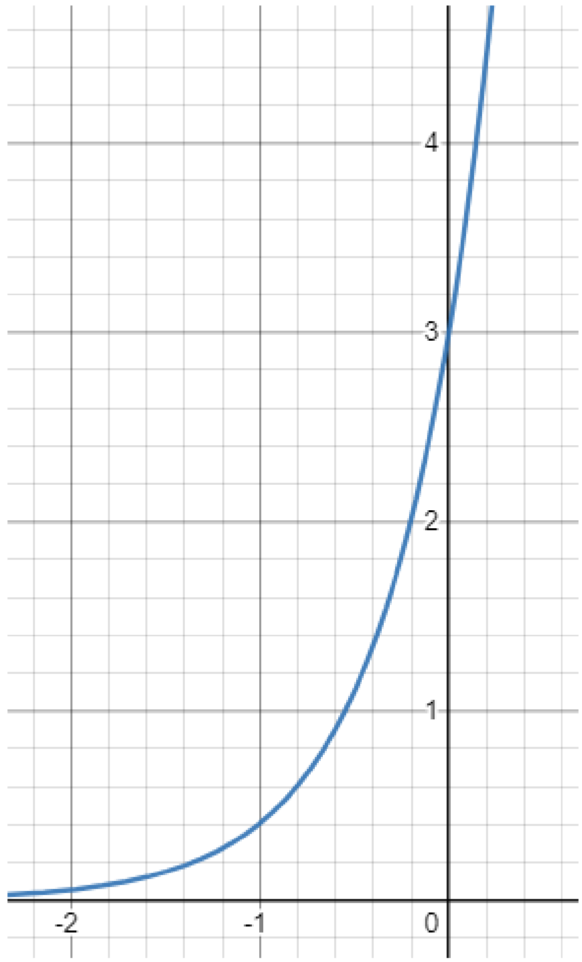
$\pm e^C$ is just an arbitrary constant, so it's ok to just relabel it as C

STEP 3: Initial Condition

$$y(0) = 3 \Rightarrow Ce^0 = 3 \Rightarrow C = 3$$

$$y(t) = 3e^{2t}$$

Interpretation: The bacteria population increases exponentially



Fact:

The general solution of $y' = ky$ is $y(t) = Ce^{kt}$

Note: From now on, please *directly* go from $y' = ky$ to $y = Ce^{kt}$ **DON'T** repeat the derivation above

Significance of C: Notice that $y(0) = Ce^{k0} = C$ so $C = y(0)$ is the initial population!

3. Solutions

What does it mean for a function to solve an ODE? It just means that if you plug in the function into the equation, both sides match:

Example 2:

Does $y = 3t + t^2$ solve $t\,(y') = y + t^2$?

$$\begin{aligned} \text{Left: } & t\,(y') = t\left(3t + t^2\right)' = t\,(3 + 2t) = 3t + 2t^2 \\ \text{Right: } & y + t^2 = 3t + t^2 + t^2 = 3t + 2t^2 \end{aligned}$$

Both sides match, and therefore the answer is **YES**

Example 3:

Does $y = t^2$ solve $y'' + ty = 2$?

$$\begin{aligned} \text{Left: } & y'' + ty = \left(t^2\right)'' + t\left(t^2\right) = 2 + t^3 \\ \text{Right: } & 2 \end{aligned}$$

The two sides don't match, and therefore the answer is **NO**

Note: Notice that both sides are equal if $t = 0$ since $2 + 0^3 = 2$, but for our ODE, we generally care whether it holds for all t, or for all $t \geq 0$

Example 4:

Determine for which r, $y = e^{rt}$ is a solution of $y'' - 5y' + 6y = 0$

$$\begin{aligned}
\left(e^{rt}\right)'' - 5\left(e^{rt}\right)' + 6\left(e^{rt}\right) &= 0 \\
r^2 e^{rt} - 5re^{rt} + 6e^{rt} &= 0 \\
\cancel{e^{rt}}\left(r^2 - 5r + 6\right) &= 0 \\
r^2 - 5r + 6 &= 0 \\
(r-2)(r-3) &= 0 \\
r = 2 \text{ or } r &= 3
\end{aligned}$$

Hence e^{2t} and e^{3t} are solutions to the ODE

4. Existence and Uniqueness

Fact:

The general solution of $y' = ky$ is $y(t) = Ce^{kt}$

Let's elaborate on this:

Example 5:

Show $y = Ce^{kt}$ solves $y' = ky$

$$\text{Left: } y' = \left(Ce^{kt}\right)' = Cke^{kt}$$
$$\text{Right: } ky = k\left(Ce^{kt}\right) = Cke^{kt} \checkmark$$

Hence $y = Ce^{kt}$ is a solution to $y' = ky$

Example 6:

Show $y = Ce^{kt}$ is the *only* solution to $y' = ky$

That is, if your friend finds a solution, they should get $y = Ce^{kt}$

Suppose y is *any* solution of $y' = ky$ consider

$$\left(ye^{-kt}\right)' = y'e^{-kt} + y\left(e^{-kt}\right)' = y'e^{-kt} + y(-k)e^{-kt} = kye^{-kt} - kye^{-kt} = 0$$

Hence the derivative of ye^{-kt} is zero

Therefore $ye^{-kt} = C$ so $y = Ce^{kt}$ ✓

Note: In this part we couldn't assume $y = Ce^{kt}$ we had to find it.

Aside: How did we get $\left(ye^{-kt}\right)'$? Notice that **IF** $y = Ce^{kt}$ then $ye^{-kt} = C$ and so $\left(ye^{-kt}\right)' = 0$. Here we reverse-engineered everything, starting from $\left(ye^{-kt}\right)' = 0$ and then obtaining $y = Ce^{kt}$

Note: On the homework, you will find a second-order analog of this. Unfortunately there is no magic recipe for dealing with those kinds of problems, it depends on the ODE at hand.

5. Appendix: Outline of the course:

Chapter 1: (4 lectures) General facts about ODE

Chapter 2: (7 lectures) First-order ODE, like $y' + t^2(y) = t^3$

Chapter 3: (6 lectures) Second-order ODE, like $y'' - 5y' + 6y = 0$

Chapter 4: (7 lectures) Laplace transform, a cool way to solve ODE

Chapter 5: (9 lectures) Systems of ODE

Chapter 6: (1 lecture) Nonlinear ODE

The interesting thing is that every chapter relates to $y' = ky$

In Chapter 2, we replace k with any function of t

In Chapter 3, we increase y' to y''

In Chapter 5, we increase y to $x(t)$ and $y(t)$