APMA 1941G – HOMEWORK 1

Problem 1: (4 points)

In the "Acoustic approximation in fluid mechanics" example from lecture, show that p^1 and $\mathbf{u}^1$ satisfy the following PDE, where

$$c_0 := \sqrt{g'(\rho^0)} \text{ and } \rho^0 \neq 0$$

$$\begin{cases} p^1_{tt} - c_0^2 \, \Delta p^1 = 0 \\ \mathbf{u}^1_{tt} - c_0^2 \nabla \left(\operatorname{div} \mathbf{u}^1 \right) = 0 \end{cases}$$

Problem 2: (16 points, 4 points each)

Suppose $u_0 = u_0(x)$ is a solution of

$$-u_0''(x) + V(x)u_0(x) = \lambda_0 u_0(x) \tag{H}$$

With $\lim_{|x|\to\infty} u_0(x) = 0$ where $x \in \mathbb{R}$, $\lambda_0 \in \mathbb{R}$, and $V(x)$ is given

Suppose we want to solve the following as a perturbation of (H)

$$-u_\epsilon''(x) + V(x)u_\epsilon(x) + \epsilon W(x)u_\epsilon(x) = \lambda_\epsilon u_\epsilon(x) \tag{H_ϵ}$$

With $\lim_{|x|\to\infty} u_\epsilon(x) = 0$ where $W = W(x)$ is given

(a) Expand u_ϵ and λ_ϵ out as

$$u_\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$
$$\lambda_\epsilon = \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \cdots$$

Plug this expansion into (H_ϵ) and show that the $O(\epsilon^k)$-terms (for $k = 1, 2, \cdots$) give you the following equation

$$-u_k'' + (V - \lambda_0)u_k = -W u_{k-1} + \sum_{j=1}^{k} \lambda_j u_{k-j} \tag{1}$$

What do you get when you compare the $O(1)-$terms?

Hint: The following formula might be useful:

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j}$$

(b) Let's look for solutions of the form

$$u_k(x) = u_0(x) w_k(x)$$

Where w_k is to be found

Plug this into (1), and multiply by u_0 and obtain

$$\left(u_0^2 w_k'\right)' = u_0^2 \left[W w_{k-1} - \sum_{j=1}^{k} \lambda_j w_{k-j}\right] \tag{2}$$

(c) Integrate (2) over $\mathbb{R}$, assuming that

$$u_0^2(x) w_k'(x) \to 0 \qquad \text{as } |x| \to \infty$$

And solve for λ_k to obtain

$$\lambda_k = \frac{\int_{-\infty}^{\infty} u_0 \left[W u_{k-1} - \sum_{j=1}^{k-1} \lambda_j u_{k-j} \right]}{\int_{-\infty}^{\infty} u_0^2}$$

This gives us a recursive definition of λ_k

(d) Integrate (2) over $(-\infty, t)$ for $t > 0$, again assuming that $u_0^2(x) w_k'(x)$ goes to 0 at $-\infty$)

Then integrate over $(-\infty, x)$ assuming $w_k(x)$ goes to 0 at $-\infty$)

And finally use $u_k(x) = u_0(x) w_k(x)$ to conclude

$$u_k(x) = u_0(x) \int_{-\infty}^{x} \frac{1}{u_0^2(t)} \int_{-\infty}^{t} u_0(s) \left[W(s) u_{k-1}(s) - \sum_{j=1}^{k} \lambda_j u_{k-j}(s) \right] ds dt.$$

This gives us a recursive definition of $u_k(x)$

This is an example of a Rayleigh-Schrödinger perturbation