LECTURE: WHAT IS AN ASYMPTOTIC EXPANSION?

Disclaimer: This course is unlike any other course that you've taken so far. You might be used to classes where the teacher states a theorem and then proves it, followed by examples. This course is more like a blur, where I throw a bunch of examples that have nothing to do with each other. This will be absolutely fascinating but also absolutely weird! Nonetheless, I do think it's a beautiful course. My speciality is PDE, which is a highly technical subject, but this course is a great way of explaining PDE without getting caught up in the technical details.

Chapter 1: Introduction

0. What is an Asymptotic Expansion?

This course is about asymptotic/perturbation methods, mostly applied to ODE/PDE

Very heuristically, the idea is this:

Suppose you have a complicated ODE/PDE that depends on a parameter ϵ, which I'll write in the form

$$A_\epsilon[u^\epsilon] = 0 \qquad (\star_\epsilon)$$

Here A_ϵ is an operator (could be nonlinear) and u^ϵ is the solution.

Example: In the PDE $\Delta u^\epsilon + \epsilon\, u^\epsilon = 0$ we have $A^\epsilon = \Delta + \epsilon I$

(1) **Question:** What happens as $\epsilon \to 0$?

In some cases, as it turns out, a much simpler structure appears in the limit, namely $u^\epsilon \to u$ in some sense, where u solves the following simpler ODE/PDE

$$A[u] = 0 \tag{$\star$}$$

Example: In example above, we hope that u^ϵ converges to a solution of $\Delta u = 0$ so $A = \Delta$

Application: This situation appears a lot in models of fluid mechanics, where they have incredibly complicated models that are impossible to study, but in some limits the models become much easier to study.

(2) Conversely, suppose you know how to solve $(\star)$

Then sometimes we can build more interesting and complicated structures as perturbations of $(\star)$

So a lot of times you'll see that we'll start from $u = u_0$ solving $(\star)$ and we build

$$u^\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \text{ (compare with a Taylor expansion)}$$

With the hope that u^ϵ solves $(\star_\epsilon)$

Goal: Make sense of this! Can we compute $u_1, u_2, \cdots$?

Analogy: If you think of u_0 as a tennis ball, u_1 might be the spin, u_2 the rotation, etc.

Disclaimer: A lot of the methods here are non-rigorous, but from time to time we'll ask ourselves: To what extent is this rigorous? When is

= an actual equality? Does this series converge?

This is analogous to Fourier series: Sure you can build a Fourier series, but it might diverge at every point!

Four Scenarios (see pictures in lecture)

Case 1: u^{ϵ} is close to u

Case 2: u^{ϵ} is close to u but oscillates wildly

Case 3: Non-uniform convergence

Case 4: u^{ϵ} doesn't converge anywhere!

The plan for the next couple of lectures (4-5 lectures) is to give you an overview of interesting examples from different applied fields

1. Example 1: Acoustic Approximation in Fluid Mechanics

Consider the following PDE from fluid mechanics

Notation:

$$x = (x_1, x_2, x_3) \text{ (space)} \qquad t > 0 \text{ (time)}$$

5 unknowns:

$$\begin{aligned} \mathbf{u} =& \mathbf{u}(x,t) = \left(u^1, u^2, u^3\right) = \text{ velocity field} \\ p =& p(x,t) = \text{ pressure} \\ \rho =& \rho(x,t) = \text{ fluid density} \end{aligned}$$

$$\begin{aligned} \operatorname{div}(\mathbf{u}) =& u^1_{x_1} + u^2_{x_2} + u^3_{x_3} \text{ (scalar)} \\ D\mathbf{u} =& \nabla \mathbf{u} = \begin{bmatrix} u^1_{x_1} & \cdots & u^1_{x_3} \\ \vdots & & \vdots \\ u^3_{x_1} & \cdots & u^3_{x_3} \end{bmatrix} \text{ (matrix)} \end{aligned}$$

Our PDE:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \text{ Conservation of Mass, scalar PDE} \tag{1}$$

$$\rho(\mathbf{u}_t + \mathbf{u}(\nabla \mathbf{u})) = -\nabla p \text{ Conservation of momentum, System} \tag{2}$$

In terms of coordinates, this means

$$\rho\left(u^i_t + \sum_{k=1}^{3} u^k u^k_{x_i}\right) = -p_{x_i}$$

Note: Here we have 4 equations, but 5 unknowns, so we need one more constraint:

Option 1: Incompressible flow: $\operatorname{div}(\mathbf{u}) = 0$

Option 2: Barotropic flow:

$$p = g(\rho) \text{ for some fixed } g \text{ with } g' > 0 \tag{3}$$

We will start with Option 2 since it makes most sense

Start with guess: Suppose you're an applied mathematician, you fiddle around with experiments, and you find that a good guess is

$$\begin{aligned} \mathbf{u}^0 =& 0 \\ \rho^0 =& \text{ constant} \\ p^0 =& \text{ constant with } p^0 = g(\rho^0) \end{aligned}$$

This is sort of like saying that, in our previous analogy, $A[u^0] = 0$ and what we want to do is to find the $A^\epsilon[u^\epsilon] = 0$ part

From this guess, let's build our perturbations

Ansatz (= formulation in German)

$$\begin{cases} \mathbf{u}^\epsilon = & \mathbf{u}^0 + \epsilon \mathbf{u}^1 + o(\epsilon) \\ \rho^\epsilon = & \rho^0 + \epsilon \rho^1 + o(\epsilon) \\ p^\epsilon = & p^0 + \epsilon p^1 + o(\epsilon) \end{cases} \tag{$\star$}$$

Here again $\mathbf{u}^0 = 0$, and ρ^0 and p^0 are given

So again, you start with the initial guess and you ask yourself: Can we perturb our guess a little bit so that it solves our PDE

Definition: $f = o(\epsilon)$ as $\epsilon \to 0$ if

$$\lim_{\epsilon \to 0} \frac{|f|}{\epsilon} = 0$$

(Think terms smaller than ϵ, like ϵ^2 or ϵ^3)

Definition: $f = O(\epsilon)$ as $\epsilon \to 0$ means $|f| \leq C\epsilon$ for small ϵ

(Think terms comparable to ϵ, like 2ϵ)

The final ingredient that we'll need is:

Fact: If $a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots = b_0 + b_1\epsilon + b_2\epsilon^2 + \cdots$ then $a_i = b_i \ \ \forall i$

(We'll prove this later in the course)

Game-Plan: Plug in our Ansatz ($\star$) into (1)-(2) and match equal powers using the Fact

Goal: Find an interesting relationship between $\mathbf{u}^0, \mathbf{u}^1, \rho^0, \rho^1, p^0, p^1$

STEP 1: Plug ($\star$) into (1)

Here we use that ρ^0 is a constant and $\mathbf{u}^0 = 0$

$$\begin{aligned}
\rho^\epsilon_t + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon) =& 0 \\
\left(\rho^0_t + \epsilon \rho^1_t + o(\epsilon)\right) + \operatorname{div}\left(\left(\rho^0 + \epsilon\rho^1 + o(\epsilon)\right)\left(0 + \epsilon \mathbf{u}^1 + o(\epsilon)\right)\right) &= 0 \\
0 + \epsilon\rho^1_t + o(\epsilon) + \operatorname{div}\left(\epsilon\rho^0\mathbf{u}^1 + \epsilon^2\rho^1\mathbf{u}^1 + o\left(\epsilon^2\right)\right) &= 0 \\
\epsilon\rho^1_t + \epsilon \operatorname{div}\left(\rho^0\mathbf{u}^1\right) + \epsilon^2 \operatorname{div}\left(\rho^1\mathbf{u}^1\right) + o(\epsilon) &= 0 = 0 + 0\epsilon + 0\epsilon^2
\end{aligned}$$

$O(\epsilon)$**-terms:**

$$\rho^1_t + \operatorname{div}\left(\rho^0\mathbf{u}^1\right) = 0$$

Since ρ^0 is a constant, we then get

$$\rho^1_t + \rho^0 \operatorname{div}\left(\mathbf{u}^1\right) = 0 \tag{4}$$

In other words, we squeezed in enough info to find a relationship between ρ^1, ρ^0 and $\mathbf{u}^1$

Let's now see if we get more info if we plug in ($\star$) into the second equation

STEP 2: Plug ($\star$) into (2)

$$\begin{aligned}
\rho^\epsilon\left(\mathbf{u}^\epsilon_t + \mathbf{u}^\epsilon\left(\nabla\mathbf{u}^\epsilon\right)\right) &= -\nabla p^\epsilon \\
\left(\rho^0 + \epsilon\rho^1 + \cdots\right)\left(\epsilon\mathbf{u}^1_t + \cdots + \left(\epsilon\mathbf{u}^1 + \cdots\right)\left(\epsilon\nabla\mathbf{u}^1 + \cdots\right)\right) &= -\nabla p^0 - \epsilon\nabla p^1 \\
\rho^0\mathbf{u}^1_t\epsilon + o(\epsilon) &= \left(-\nabla p^1\right)\epsilon + o(\epsilon)
\end{aligned}$$

Here we used that $\mathbf{u}_0 = 0$ and that p_0 is constant

Comparing the $O(\epsilon)$ terms we get

$$\rho^0 \mathbf{u}_t^1 = -\nabla p^1 \tag{5}$$

Note: In this example, we only looked at the $O(\epsilon)$ terms, but sometimes (like on the HW) you have to consider higher-order terms, like $O(\epsilon^2), O(\epsilon^3)$

Equations (4) and (5) seem like a dead-end but in math it's important to persevere! Some things may look like a dead-end but if you work hard enough, you may find a way out!

In fact, there is one info that we haven't used yet! Namely our barytropic condition (3)

$$p = g(\rho)$$

STEP 3: Plug ($\star$) into (3)

$$\begin{aligned} p^\epsilon &= g\left(\rho^\epsilon\right) \\ p^0 + \epsilon p^1 + \cdots &= g\left(\rho^0 + \epsilon\rho^1 + \cdots\right) \\ &= g(\rho^0) + \epsilon\rho^1 g'(\rho^0) + o(\epsilon) \quad \text{(Taylor expansion)} \end{aligned}$$

Comparing the $O(\epsilon)$ terms we get

$$p^1 = g'(\rho^0)\rho^1 \tag{6}$$

One final ingredient: We have 3 new equations now (4), (5), (6) and we want to see how to make them meaningful

STEP 4: Go back to (4)

$$\rho_t^1 + \rho^0 \operatorname{div}\left(\mathbf{u}^1\right) = 0$$

Notice that (5) involves $\mathbf{u}_t^1$ so perhaps it's useful to differentiate (4) with respect to t:

$$\begin{aligned}
\rho_{tt}^1 &= -\rho^0 \operatorname{div}\left(\mathbf{u}_t^1\right) \\
&= -\operatorname{div}\left(\rho^0 \mathbf{u}_t^1\right) \qquad \rho^0 \text{ is constant} \\
&= -\operatorname{div}\left(-\nabla p^1\right) \quad \text{By (5)} \\
&= \operatorname{div}\left(\nabla p^1\right) \\
&= \Delta p^1 \qquad\qquad (\text{ because } \operatorname{div}(\nabla p^1) = \sum_{i=1}^{3} p_{x_i x_i}^1 = \Delta p^1) \\
&= \Delta\left(g'(\rho^0)\rho^1\right) \qquad \text{By (6)} \\
&= g'(\rho^0)\Delta\rho^1
\end{aligned}$$

Therefore, if you let

$$c_0 =: \sqrt{g'(\rho^0)}$$

We then get

$$\rho_{tt}^1 = (c_0)^2\, \Delta\rho^1$$

So in fact the first-order correction ρ^1 solves a wave equation!

Note: On the HW you will show that

$$\begin{aligned}
p_{tt}^1 &= (c_0)^2\, \Delta p^1 \\
\mathbf{u}_{tt}^1 &= (c_0)^2\, \nabla\left(\operatorname{div}\mathbf{u}^1\right)
\end{aligned}$$

This is precisely why it's called an acoustic approximation, because the first-order corrections solve wave equations.

In practice, we can measure c_0 from that