1 Research Related to Teaching

I would first like to mention research related to teaching, since it is more relevant to your application. While I haven't published any concrete results, my research has been shifting more towards teaching-related research. In particular, I am interested in the following questions:

1.1 YouTube and Learning

Since I have a YouTube channel called Dr Peyam, I would like to figure out how to use YouTube videos to promote learning both inside and outside the classroom. So far, I have found at least 10 benefits:

(1) Great if you want to make extra review problems before the exam. This is what I did when I taught Math 2E (Vector Calculus) at UC Irvine. For each topic covered in the course (e.g. Stokes Theorem), I made a video covering a sample problem on that topic.

(2) You can use it to create 'snippets' of your lecture. For instance, if in Linear Algebra you talk about Cramer's Rule, you can make a 10 minute video covering that topic. It's useful both for people who want to review what happened in lecture, and for people who want to preview the lecture.

(3) You can use it to make videos about topics you didn't have time to talk about in lecture. This allows you to really focus on the essentials when lecturing, and you can do more in-class activities that way without worrying about covering every single topic

(4) Especially for upper-division math courses, in lecture you can give the main idea of the proof of a theorem, and then make a video where you fill in the gaps of a proof. This is especially useful for lengthy proofs like the Replacement Theorem in Linear Algebra

(5) If you don't have time to hold a review session, you can make virtual review session videos (very similar to the first point)

(6) You can hold virtual office hours via the YouTube livestream option, where, at a predetermined time, you send a link to the students. Once the students click on the link, they get virtually teleported to your office hours and can send

questions via YouTube chat. This is great for students who cannot physically make it to your office hours because of commuting; and even those who cannot attend can always watch the transcript of what happened.

(7) Students who are especially shy during class can comment on the videos you make with questions

(8) You can actually check how many students watch your video by ‘unlisting’ it, which means only those with a link can watch it

(9) It’s an excellent outreach opportunity; you’re not only making videos for your students, but also for the rest of the world

(10) Some future plans: See how to use the videos during lecture. Maybe ask the students to pre-watch a video that presents a problem and then immediately start class with the solution of that problem.

Some of my planned research will focus on finding more advantages and finding more ways to incorporate the videos during my lectures. Also, it would be great to figure out how useful those online resources are by comparing the students’ performance with lectures that don’t use them. Finally, I would like to mention that I have presented this idea at the Fall 2019 MAA SoCal Nevada Sectional Meeting, the 2020 Joint Math Meetings, and the Fall 2020 ArizMATYC conference, and it’s been well-received.

1.2 Active Learning

After learning about the benefits of active learning at the conference above, I have been recently trying to change my teaching strategy: Instead of just lecturing for the whole hour, I would write a problem on the board and ask them to discuss it with each other, before I present the solution. I am interested in figuring out how effective this strategy is, and whether it helps the students absorb the material. So far, I have seen only positive results; the students in my Linear Algebra class this quarter were definitely livelier, and they did better on the midterm compared to the students in the same course last winter. It seems to work better when there are more students involved than fewer ones, that’s why I think this might be a great strategy for large lecture classes. In the future, I would like to find more such strategies and measure their effectiveness.

2 Past research: Chemical Reactions & Diffusions

2.1 The Kramers-Smoluchowski equation

Introduction My research lies primarily in the field of partial differential equations (PDE) and the calculus of variations. I am interested in studying various asymptotic limits that arise in PDE, like the following one, which arises in the study of chemical reactions:

Chemical Context Consider a simple chemical reaction $A \rightleftharpoons B$, where a molecule A transforms into B and vice-versa. Then there are two ways of viewing this reaction:

Macroscopic Level There are two states A and B, and we denote by $\alpha = \alpha(x,t)$ (resp. β) the concentration of A (resp. B) at x and time t. Then α and β solve the following

Reaction-Diffusion System

$$\begin{cases} \alpha_t - \Delta_x \alpha = \kappa(\beta - \alpha) \\ \beta_t - \Delta_x \beta = \kappa(\alpha - \beta) \end{cases} \tag{R-D}$$

Here $\kappa > 0$ is the reaction-rate constant of the reaction $A \rightleftharpoons B$. One of the problems in chemistry to characterize κ in terms of the parameters of the system.

Microscopic Level Instead of just 2 states A and B, you can think of the molecule having a continuum of states, parametrized by a chemical variable $\xi \in \mathbb{R}$, where $\xi = -1$ corresponds to A and $\xi = 1$ to B. Then the molecule can be thought of as a Brownian particle traveling on a double-well potential function $\Phi = \Phi(\xi)$, as in Figure $1(a)$ on the next page. Here Φ is an even function, normalized so that $\Phi(0) = 1, \Phi(\pm 1) = 0, \Phi(\pm 2) = 1$. In this case, the dynamics of the reaction are given by a Stochastic Differential Equation (SDE), whose Fokker-Planck equation is the following (normalized)

Kramers-Smoluchowski equation

$$\sigma^\epsilon \left(u_t^\epsilon - \Delta_x u^\epsilon \right) = \left(\frac{\sigma^\epsilon}{\tau_\epsilon} u_\xi^\epsilon \right)_\xi \tag{KS$_\epsilon$}$$

Here $u^\epsilon = u^\epsilon(x,\xi,t)$ is the density function of the particle, $\tau_\epsilon = \frac{1}{\epsilon^2} e^{-\frac{1}{\epsilon^2}}$ is a scaling factor that gives a nontrivial limit, and $\sigma^\epsilon = C_\epsilon e^{-\frac{\Phi}{\epsilon^2}}$, where C_ϵ makes $\int_{\mathbb{R}} \sigma^\epsilon = 1$.

Main Result. *For all* $0 \leq t \leq T$*, as* $\epsilon \to 0$*, we have:*

$$\sigma^\epsilon u^\epsilon \rightharpoonup \alpha\, \delta_{\{\xi=-1\}} + \beta\, \delta_{\{\xi=1\}}$$

for some α *and* β*. More importantly,* α *and* β *solve* (R-D) *with* $\kappa = \frac{\sqrt{|\Phi''(0)|\Phi''(1)}}{2\pi}$.

This result shows that (KS_ϵ) and (R-D) are just two different sides of the same coin. Moreover, it characterizes the reaction-rate constant κ in terms of our potential function Φ, which is just what we want.

Idea of proof This result has already been proven by Peletier, Savaré, and Veneroni in [PSV12], using $\Gamma-$convergence, as well as by Herrmann and Niethammer, where they rewrite (KS_ϵ) as a gradient flow on the Wasserstein space of probability measures and using a Rayleigh-type dissipation functional, and finally by Arnrich et. al. where they interpret the dynamics as a curve of maximal slope in a Wasserstein space. In my thesis [Tab16] and in joint work with my PhD advisor Lawrence C. Evans [ET16], we provide a much simpler and more direct proof, requiring just a simple integration by parts. The main idea is to devise a test function ϕ^ϵ which, after multiplying (KS_ϵ) by ϕ^ϵ and integrating by parts, cancels out the singular term σ^ϵ in (KS_ϵ):

$$\phi^\epsilon(\xi) = \int_0^{\Lambda(\xi)} \frac{\tau_\epsilon}{\sigma^\epsilon} d\xi \quad \text{where } \Lambda(s) = \begin{cases} -3/2 & \text{if} \quad s \leq -3/2 \\ s & \text{if} \quad -3/2 \leq s \leq 3/2 \\ 3/2 & \text{if} \quad s \geq 3/2 \end{cases}$$

The proof is robust enough that we can modify it to treat more general cases. All proofs rely on building test functions similar to ϕ^ϵ above.

2.2 Generalizations

Three wells If Φ has three wells at $-2, 0, 2$ as in Figure $1(b)$ above, then $\rho^\epsilon \rightharpoonup \alpha\, \delta_{-2} + \beta\, \delta_0 + \gamma\, \delta_2$, where α, β, δ solve

$$\begin{cases} \alpha_t - \Delta_x \alpha = \kappa\,(\beta - \alpha) \\ \beta_t - \Delta_x \beta = \kappa\,(\alpha - 2\beta + \gamma) \\ \gamma_t - \Delta_x \gamma = \kappa\,(\beta - \gamma) \end{cases} \tag{R-D}$$

Infinitely many wells If $\Phi(2m) = 0$ for $m \in \mathbb{Z}$, as in Figure 1(c) above, then modifying Z_ϵ so that $\int_{-1}^{1} \sigma^\epsilon d\xi = 1$, we get that $\rho^\epsilon \rightharpoonup \sum_{m=-\infty}^{\infty} \alpha^m \delta_{2m}$ for functions α^m ($m \in \mathbb{Z}$), which satisfy the infinite system

$$\alpha_t^m - \Delta_x \alpha^m = 2\kappa \left(\alpha^{m-1} - 2\alpha^m + \alpha^{m+1}\right)$$

2.3 Higher-dimensional case

We were able to generalize the result to the case where the chemical variable ξ is multi-dimensional. Assume that $\Phi : \mathbb{R}^m \to \mathbb{R}$ is smooth, nonnegative, even in the first variable ξ_1, has two wells at the points $e^\pm = (\pm 1, 0, \cdots, 0)$, normalized so that $\Phi(0) = 1$, $\Phi(e^\pm) = 0$, and moreover $\det D^2\Phi(e^\pm) \neq 0$ and $D^2\Phi(0)$ is diagonal, with eigenvalues $\lambda_1 < 0 < \lambda_2 \leq \cdots \leq \lambda_m$. (see Figure 2 below) Then the analog of (KS$_\epsilon$) reads as:

$$\tau_\epsilon \sigma^\epsilon \left(u_t^\epsilon - \Delta_x u^\epsilon\right) = \mathrm{div}_\xi \left(\sigma^\epsilon \nabla_\xi u^\epsilon\right)$$

And we obtain that $\rho^\epsilon \rightharpoonup \alpha \delta_{e^-} + \beta \delta_{e^+}$, where α, β solve:

$$\begin{cases} \alpha_t - \Delta_x \alpha = \kappa \left(\beta - \alpha\right) \\ \beta_t - \Delta_x \beta = \kappa \left(\alpha - \beta\right) \end{cases}$$

Here the reaction-rate constant κ is inspired by capacity estimates from [BEGK04]

$$\kappa = \frac{|\lambda_1|}{2\pi} \frac{\sqrt{|\det D^2\Phi(e^\pm)|}}{\sqrt{|\det D^2\Phi(0)|}}$$

In this case, we construct our test-function ϕ^ϵ via a PDE:

$$-\,\mathrm{div}\left(\frac{\sigma^\epsilon}{\tau_\epsilon} \nabla \phi^\epsilon\right) = \frac{1}{|B^+|}\chi_{B+} - \frac{1}{|B^-|}\chi_{B-}$$

where $B^\pm =: \left\{\Phi(\xi) \leq \frac{1}{4}\right\} \cap \mathbb{R}^m_\pm$, as in Figure 2.
In recent joint work with Insuk Seo [ST20], we were able to prove the result for more general Φ, without any symmetry assumptions, and further allow Φ to have multiple wells. We achieved that by interpreting the wells as equilibrium states of a Markov process and proving some results about metastable processes, which are processes with multiple stable equilibria. I would also like to mention that this generalized result has been proven independently by Michel and Zworski [MZ18] using semiclassical analysis techniques.

2.4 Open Problems

2.4.1 Revolution Well

Now take the one-dimensional well in the $yz-$plane, and rotate it around the $z-$axis to obtain a hat-like figure, as in Figure $3(a)$ below. One can then show that the solutions concentrate on the unit circle $x^2 + y^2 = 1$ in the $xy-$plane, but it is not clear what the resulting equations would look like.

2.4.2 Translation-well

This time, take the double-well Φ in the $yz-$plane and just translate it along the $x-$axis, as in Figure $3(b)$ below. Again, the solution concentrates along the lines $y = \pm 1$, but it is not clear what the resulting equation looks like.

2.4.3 Limit of wells

(Suggested by Yu Ding at CSULB) Suppose $\Phi = \Phi^n$ has n wells, each spaced $\frac{1}{n}$ apart, as in Figure $3(c)$ below. First take the limit of u^ϵ as ϵ goes to 0 and then the limit as n goes to ∞. Does the resulting limit exist? And if yes, what does it look like? This would resemble the passage in statistical mechanics from a discrete limit to a continuous limit.

2.4.4 The nonlinear case

Consider the following nonlinear analog of (KS_ϵ)

$$\sigma^\epsilon u^\epsilon_t - \Delta u^\epsilon = \left(\frac{\sigma^\epsilon}{\tau_\epsilon} f(u^\epsilon_\xi) \right)_\xi$$

In this case f satsifies a monotonicity condition $xf(x) \geq C\left|x\right|^2$ for some $C > 0$. Although the convergence results are the same, it is not clear what the resulting PDE looks like. In the linear case, we have a representation formula for u^ϵ:

$$u^\epsilon(x,\xi,t) = \kappa(\beta(x,t) - \alpha(x,t)) \left(\int_0^\xi \frac{\tau_\epsilon}{\sigma^\epsilon} d\xi \right) + (\alpha(x,t) + \beta(x,t)),$$

Moreover, as suggested in [PSV12] in the linear case, perhaps the stationary solution of (K-S) is useful, which is here

$$u^{\epsilon}(x,\xi,t) = \int_0^{\xi} f^{-1}\left(A(x,t)\frac{\tau_\epsilon}{\sigma^{\epsilon}(\xi)}\right) d\xi + B(x,t)$$

3 Current Work: Kolmogorov-Type Flow

In addition to the chemical reaction model above, I am currently investigating a KAM flow that exhibits interesting behavior.

Consider the following dynamical system, where $x = x(t), y = y(t), z = z(t)$:

$$\begin{cases} x' = \sin(z) \\ y' = \sin(x) \\ z' = \sin(y) \end{cases}$$

This is an example of a Kolmogorov-type flow, and is mentioned for instance in [JGP92] as a substitute of a fast dynamos, which are used to model astrophysical magnetic fields. Although this flow exhibits chaotic behavior, I nevertheless show that, near the initial condition $\left(\pi, \frac{\pi}{2}, 0\right)$, there exists a quasi-periodic solution

Theorem. *There exist $x_0 > 0$ and $t_0 > 0$ such that, if $X(t;x_0)$ is the solution of with initial conditions $\left(\pi + x_0, \frac{\pi}{2}, 0\right)$, then for all $t \geq 0$,*

$$X(t + t_0; x_0) = X(t; x_0) + (0, 0, 2\pi)$$

The main idea of the proof is to show that, for some x_0, the solution $X(t;x_0)$ must cross the segment $\{x = 2\pi\} \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left\{z = \frac{\pi}{2}\right\}$ (see Figure 4; The solution must cross the horizontal line) and then using a symmetry argument à la [XYZ16].

In order to show the former, we use a computer-assisted proof using an intermediate value-type argument. I further show that the flow is chaotic by (numerically) calculating the Floquet exponents and showing that they're outside the unit disk.

Some open questions: It would be interesting to see if the above property generalizes to the following variations of the above flow, such as replacing sin by any 2π periodic function f with mean 0. Another (one suggested by Jack Xin at UCI) is a nice variation of the classical ABC flow in [XYZ16]

$$\begin{cases} x' = \sin(z) - \cos(y) \\ y' = \sin(x) - \cos(z) \\ z' = \sin(y) - \cos(x) \end{cases}$$

4 Future project: The G-equation

I am planning on investigating more PDE-related questions. The G-equation is a PDE that models turbulent combustion and in particular is used to model flame propagation [OF03, Pet00]

$$-\operatorname{div}\left(\frac{\nabla G^\epsilon}{|\nabla G^\epsilon|}\right)|\nabla G^\epsilon| + |\nabla G^\epsilon| + AV\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right) \cdot \nabla G^\epsilon = 0$$

Here $A > 0$ is the flow intensity and $V(x,y) = (-H_y, H_x)$ where $H(x,y) = \sin(2\pi x)\sin(2\pi y)$

Question: Does the limit of G^ϵ as $\epsilon \to 0$ exist when A is large? That is, does the flow homogenize?

Homogenization here is similar in spirit to the passage from the microscopic case (when $\epsilon > 0$) to the macroscopic case (when $\epsilon = 0$). When A is small, then the first-order part is coercive and homogenization follows from Caffarelli and Monneau's work [CM14]. In that paper, they also provide a counterexample to homogenization by constructing an appropriate barrier function, and it would be interesting to see if one could use that example to disprove homogenization if A is large enough.

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C:/Users/Peyam Tabrizian/OneDrive/Irvine/Applications/Phi.png

Figure 1: Double-well potential and variations

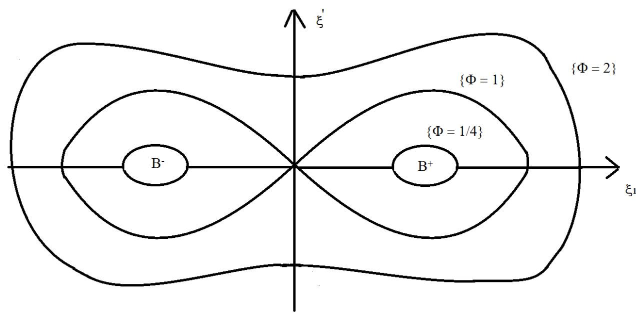

Figure 2: Level sets of Φ

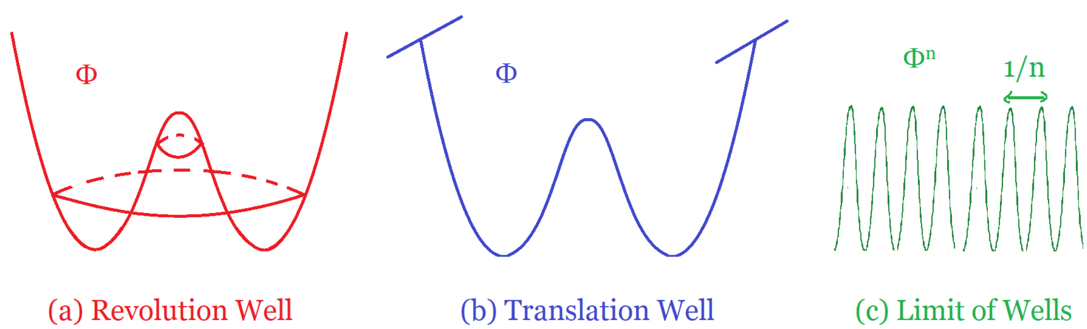

Figure 3: Some Open Problems

Figure 4: The solution must cross the horizontal line