APMA 0350 – FINAL EXAM – SOLUTIONS

1.

$$\begin{cases} \dfrac{dy}{dx} = -\left(\dfrac{3x^2y+2y^3}{x^3+6xy^2}\right) \\ y(1) = 2 \end{cases}$$

STEP 1:

$$\begin{aligned} \frac{dy}{dx} &= -\left(\frac{3x^2y+2y^3}{x^3+6xy^2}\right) \\ \left(x^3+6xy^2\right) dy &= -\left(3x^2y+2y^3\right) dx \\ \left(3x^2y+2y^3\right) dx + \left(x^3+6xy^2\right) dy &= 0 \end{aligned}$$

STEP 2: Check exact

$$\begin{aligned} P_y &= \left(3x^2y+2y^3\right)_y = 3x^2+6y^2 \\ Q_x &= \left(x^3+6xy^2\right) = 3x^2+6y^2 \end{aligned}$$

Since $P_y = Q_x$ the ODE is exact ✓

STEP 3:

$$\langle P, Q\rangle = \nabla f = \langle f_x, f_y\rangle$$

$$f_x = P \Rightarrow f(x,y) = \int 3x^2y+2y^3 dx = x^3y+2xy^3+g(y)$$

$$f_y = Q \Rightarrow f(x,y) = \int x^3+6xy^2 dy = x^3y+6x\left(\frac{y^3}{3}\right)+h(x) = x^3y+2xy^3+h(x)$$

$$f(x,y) = x^3y+2xy^3$$

STEP 3: Solution:

$$x^3y+2xy^3 = C$$

STEP 4: Initial Condition:

$$\begin{aligned} y(1) =& 2 \\ (1)^3 2 + 2(1)2^3 =& C \\ 2 + 16 =& C \\ C =& 18 \end{aligned}$$

STEP 5: Answer:

$$x^3 y + 2xy^3 = 18$$

2.

$$\begin{cases} t^4\left(y'\right) + t^3 y = t^4 \sin(t) \\ y(\pi) = 2 \end{cases}$$

STEP 1: Standard form

$$y' + \left(\frac{1}{t}\right) y = \sin(t)$$

STEP 2: Integrating factors

$$e^{\int 1 t dt} = e^{\ln(t)} = t$$

STEP 3: Multiply by t

$$\begin{aligned}
t\left(y' + \frac{1}{t}y\right) &= t\sin(t) \\
(ty)' &= t\sin(t) \\
ty &= \int t\sin(t)dt \\
ty &= -t\cos(t) - \int (1)\left(-\cos(t)\right)dt \qquad \text{(Integration by parts)} \\
ty &= -t\cos(t) + \int \cos(t)dt \\
ty &= -t\cos(t) + \sin(t) + C \\
y &= -\cos(t) + \left(\frac{\sin(t)}{t}\right) + \left(\frac{C}{t}\right)
\end{aligned}$$

STEP 4: Initial Condition

$$\begin{aligned}
y(\pi) &= 2 \\
-\cos(\pi) + \left(\frac{\sin(\pi)}{\pi}\right) + \frac{C}{\pi} &= 2 \\
-(-1) + 0 + \frac{C}{\pi} &= 2 \\
1 + \frac{C}{\pi} &= 2 \\
\frac{C}{\pi} &= 1 \\
C &= \pi
\end{aligned}$$

STEP 5: Answer

$$y = -\cos(t) + \left(\frac{\sin(t)}{t}\right) + \left(\frac{\pi}{t}\right)$$

3. **STEP 1:**

$$
\begin{aligned}
y'' - 3y' - 4y &= 5e^{4t} \\
\left(D^2 - 3D - 4\right) y &= 5e^{4t} \\
(D+1)\underbrace{(D-4)\,y}_{z} &= 5e^{4t} \\
(D+1)\,z &= 5e^{4t} \qquad \text{where } z = (D-4)y
\end{aligned}
$$

STEP 2:

$$
\begin{aligned}
(D+1)z &= 5e^{4t} \\
z' + z &= 5e^{4t} \\
e^{t}\left(z' + z\right) &= e^{t}5e^{4t} \qquad \text{Integrating Factors} \\
\left(e^{t}z\right)' &= 5e^{5t} \\
e^{t}z &= \int 5e^{5t}dt = e^{5t} + A \\
z &= e^{4t} + Ae^{-t}
\end{aligned}
$$

STEP 3:

$$
\begin{aligned}
z =& e^{4t} + Ae^{-t} \\
(D-4)y =& e^{4t} + Ae^{-t} \\
y' - 4y =& e^{4t} + Ae^{-t} \\
e^{-4t}(y' - 4y) =& e^{-4t}e^{4t} + e^{-4t}Ae^{-t} \qquad \text{Integrating Factors} \\
\left(e^{-4t}y\right)' =& 1 + Ae^{-5t} \\
e^{-4t}y =& \int 1 + Ae^{-5t} dt \\
e^{-4t}y =& t + A\left(\frac{e^{-5t}}{-5}\right) + B \\
e^{-4t}y =& t - \left(\frac{A}{5}\right)e^{-5t} + B \\
y =& te^{4t} - \left(\frac{A}{5}\right)e^{-5t}e^{4t} + Be^{4t} \\
y =& te^{4t} - \left(\frac{A}{5}\right)e^{-t} + Be^{4t}
\end{aligned}
$$

STEP 4: Answer

$$
y = te^{4t} - \left(\frac{A}{5}\right)e^{-t} + Be^{4t} = te^{4t} + Ae^{-t} + Be^{4t}
$$

Note: Here is the solution using undetermined coefficients:

STEP 1: Homogeneous Solution: $y'' - 3y' + 4y = 0$

Aux: $r^2 - 3r + 4 = 0 \Rightarrow (r+1)(r-4) = 0 \Rightarrow r = -1, 4$

$$
y_0 = Ae^{-t} + Be^{4t}
$$

STEP 2: Particular Solution:

$5e^{4t} \rightsquigarrow r = 4$ which coincides so we guess $y_p = Ate^{4t}$

$$
\begin{aligned}
(y_p)' =& Ae^{4t} + 4Ate^{4t} \\
(y_p)'' =& 4Ae^{4t} + 4Ae^{4t} + (4At)\,4e^{4t} = 8Ae^{4t} + 16Ate^{4t}
\end{aligned}
$$

$$\begin{aligned}
(y_p)'' - 3(y_p)' - 4(y_p) &= 5e^{4t} \\
8Ae^{4t} + 16Ate^{4t} - 3\left(Ae^{4t} + 4Ate^{4t}\right) - 4Ate^{4t} &= 5e^{4t} \\
8Ae^{4t} + \cancel{16Ate^{4t}} - 3Ae^{4t} - \cancel{12Ate^{4t}} - \cancel{4Ate^{4t}} &= 5e^{4t} \\
5Ae^{4t} &= 5e^{4t} \\
A &= 1
\end{aligned}$$

Therefore $y_p = te^{4t}$

STEP 3: General Solution

$$y = y_0 + y_p = Ae^{-t} + Be^{4t} + te^{4t}$$

4.

$$\begin{cases} y'' = \lambda y \\ y'(0) = 0 \\ y(6) = 0 \end{cases}$$

STEP 1: Auxiliary Equation: $r^2 = \lambda$

Case 1: $\lambda > 0$

Then $r^2 = \lambda = \omega^2$ and so $r = \pm\omega$

$$y(t) = Ae^{\omega t} + Be^{-\omega t}$$
$$y'(t) = A\omega e^{\omega t} - B\omega e^{-\omega t}$$
$$y'(0) = A\omega - B\omega = 0 \Rightarrow A\omega = B\omega \Rightarrow B = A$$
$$y(t) = Ae^{\omega t} + Ae^{-\omega t}$$
$$y'(t) = A\omega e^{\omega t} - A\omega e^{-\omega t}$$

$$y(6) = 0$$
$$\not{A}e^{3\omega} + \not{A}e^{-6\omega} = 0$$
$$e^{6\omega} + e^{-6\omega} = 0$$

But this is a contradiction since the left is > 0 $\Rightarrow\Leftarrow$

Case 2: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (repeated twice)

$$y(t) = A + Bt$$

$$y'(t) = B$$
$$y'(0) = B = 0 \Rightarrow B = 0$$
$$y(t) = A$$
$$y(6) = 0 \Rightarrow A = 0$$

But then $y = 0 \Rightarrow\Leftarrow$

Case 3: $\lambda < 0$

In this case $\lambda = -\omega^2$ where $\omega > 0$

Aux: $r^2 = \lambda = -\omega^2 \Rightarrow r = \pm\omega i$

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$y'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$y'(0) = -A\omega\sin(0) + B\omega\cos(0) = B\omega = 0 \Rightarrow B = 0$$

$$y(t) = A\cos(\omega t)$$

$$\begin{aligned} y(6) &= 0 \\ \not{A}\cos(6\omega) &= 0 \\ \cos(6\omega) &= 0 \\ 6\omega &= \frac{\pi}{2} + \pi m \\ \omega &= \frac{\pi}{12} + \left(\frac{\pi}{6}\right) m \end{aligned}$$

Here we start at $m = 0$ since $\omega > 0$ if $m = 0$

STEP 2: Answer:

Eigenvalues:

$$\lambda = -\omega^2 = -\left(\frac{\pi}{12} + \left(\frac{\pi}{6}\right) m\right)^2 \qquad m = 0, 1, 2, \cdots$$

Eigenfunctions:

$$y = \cos(\omega t) = \cos\left(\left(\frac{\pi}{12} + \left(\frac{\pi}{6}\right) m\right) t\right) \qquad m = 0, 1, 2 \cdots$$

5.

$$\begin{cases} y'' + 3y' + 2y = & 12e^{2t} \\ y(0) = & 4 \\ y'(0) = & 2 \end{cases}$$

STEP 1: Take Laplace transforms

$$\begin{aligned} \mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} =& \mathcal{L}\left\{12e^{2t}\right\} \\ \left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} =& \frac{12}{s-2} \\ s^2\mathcal{L}\{y\} - 4s - 2 + 3s\mathcal{L}\{y\} - 3(4) + 2\mathcal{L}\{y\} =& \frac{12}{s-2} \\ \left(s^2 + 3s + 2\right)\mathcal{L}\{y\} - 4s - 14 =& \frac{12}{s-2} \end{aligned}$$

$$\mathcal{L}\{y\} = \frac{4s+14}{s^2+3s+2} + \frac{12}{(s-2)(s^2+3s+2)}$$

STEP 3: Partial Fraction 1

$$\begin{aligned} \frac{4s-14}{s^2+3s+2} =& \frac{4s+14}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \\ =& \frac{A(s+2) + B(s+1)}{(s+1)(s+2)} \\ =& \frac{(A+B)s + (2A+B)}{s^2+3s+2} \end{aligned}$$

$$\begin{cases} A + B = & 4 \\ 2A + B = & 14 \end{cases}$$

Subtracting the second equation from the first we get $A = 10$ and then $B = 4 - A = 4 - 10 = -6$

$$\frac{4s-14}{s^2+3s+2} = \frac{10}{s+1} - \frac{6}{s+2}$$

STEP 4: Partial Fraction 2

$$
\begin{aligned}
\frac{12}{(s-2)(s^2+3s+2)} =& \frac{12}{(s-2)(s+1)(s+2)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s+2} \\
=& \frac{A(s+1)(s+2) + B(s-2)(s+2) + C(s-2)(s+1)}{(s-2)(s+1)(s+2)} \\
=& \frac{As^2 + 3As + 2A + Bs^2 - 4B + Cs^2 - Cs - 2C}{(s-2)(s^2+3s+2)} \\
=& \frac{(A+B+C)s^2 + (3A-C)s + (2A-4B-2C)}{(s-2)(s^2+3s+2)}
\end{aligned}
$$

$$
\begin{cases}
A + B + C =& 0 \\
3A - C =& 0 \\
2A - 4B - 2C =& 12
\end{cases}
$$

The second equation gives $C = 3A$ and the first equation then gives $A + B + 3A = 0 \Rightarrow B + 4A = 0 \Rightarrow B = -4A$.

And finally plugging into the third equation we get

$$
\begin{aligned}
2A - 4B - 2C =& 12 \\
2A - 4(-4A) - 2(3A) =& 12 \\
2A + 16A - 6A =& 12 \\
12A =& 12 \\
A =& 1
\end{aligned}
$$

And $B = -4A = -4$ and $C = 3A = 3$ and so

$$
\begin{cases}
A =& 1 \\
B =& -4 \\
C =& 3
\end{cases}
$$

$$
\frac{12}{(s-2)(s^2+3s+2)} = \frac{1}{s-2} - \frac{4}{s+1} + \frac{3}{s+2}
$$

STEP 5:

$$\begin{aligned}
\mathcal{L}\left\{y\right\} =& \frac{4s-14}{s^2+3s+2} + \frac{12}{(s-2)(s^2+3s+2)} \\
=& \left(\frac{10}{s+1} - \frac{6}{s+2}\right) + \left(\frac{1}{s-2} - \frac{4}{s+1} + \frac{3}{s+2}\right) \\
=& \frac{6}{s+1} - \frac{3}{s+2} + \frac{1}{s-2} \\
=& \mathcal{L}\left\{6e^{-t} - 3e^{-2t} + e^{2t}\right\}
\end{aligned}$$

STEP 6: Answer:

$$y = 6e^{-t} - 3e^{-2t} + e^{2t}$$

6.

$$\phi'(t) + 5\int_0^t e^{2(t-\tau)}\phi(\tau)d\tau = 0 \quad \text{with } \phi(0) = 6$$

STEP 1: This is of the form

$$\phi'(t) + 5\left(e^{2t} \star \phi\right) = 0$$

STEP 2: Take Laplace transforms

$$\begin{aligned}
\mathcal{L}\{\phi'(t)\} + 5\mathcal{L}\left\{e^{2t} \star \phi\right\} =& \mathcal{L}\{0\} \\
s\mathcal{L}\{\phi\} - \phi(0) + 5\mathcal{L}\left\{e^{2t}\right\}\mathcal{L}\{\phi\} =& 0 \\
s\mathcal{L}\{\phi\} - 6 + \left(\frac{5}{s-2}\right)\mathcal{L}\{\phi\} =& 0 \\
\left(s + \frac{5}{s-2}\right)\mathcal{L}\{\phi\} =& 6 \\
\frac{s(s-2)+5}{s-2}\mathcal{L}\{\phi\} =& 6 \\
\frac{s^2-2s+5}{s-2}\mathcal{L}\{\phi\} =& 6 \\
\mathcal{L}\{\phi\} =& \frac{6(s-2)}{s^2-2s+5}
\end{aligned}$$

STEP 3: Complete the square

$$\begin{aligned}
\frac{6(s-2)}{s^2-2s+5} =& \frac{6(s-2)}{(s-1)^2-1+5} = \frac{6(s-2)}{(s-1)^2+4} = \frac{6(s-1-1)}{(s-1)^2+4} \\
=& \frac{6(s-1)}{(s-1)^2+4} - \frac{6}{(s-1)^2+4}
\end{aligned}$$

This is a shifted version by 1 unit of

$$\frac{6s}{s^2+4} - \frac{6}{s^2+4} = 6\cos(2t) - 3\sin(2t)$$

$$\text{Therefore } \frac{6(s-2)}{s^2-2s+5} = \mathcal{L}\left\{e^t\left(6\cos(2t) - 3\sin(2t)\right)\right\}$$

STEP 4: Answer

$$\phi(t) = 6e^t\cos(2t) - 3e^t\sin(2t)$$

7.

$$\mathbf{x}' = A\mathbf{x} \qquad A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\left|A - \lambda I\right| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - (-1) = (\lambda+2)^2 + 1 = 0$$

Which gives $(\lambda+2)^2 = -1 \Rightarrow \lambda + 2 = \pm i \Rightarrow \lambda = -2 \pm i$

STEP 2: $\lambda = -2 + i$

$$\begin{aligned}
\text{Nul } (A - (-2+i)I) &= \left[\begin{array}{cc|c} -2-(-2+i) & 1 & 0 \\ -1 & -2-(-2+i) & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|c} -1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$x + iy = 0 \Rightarrow x = -iy$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -iy \\ y \end{bmatrix} = y \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

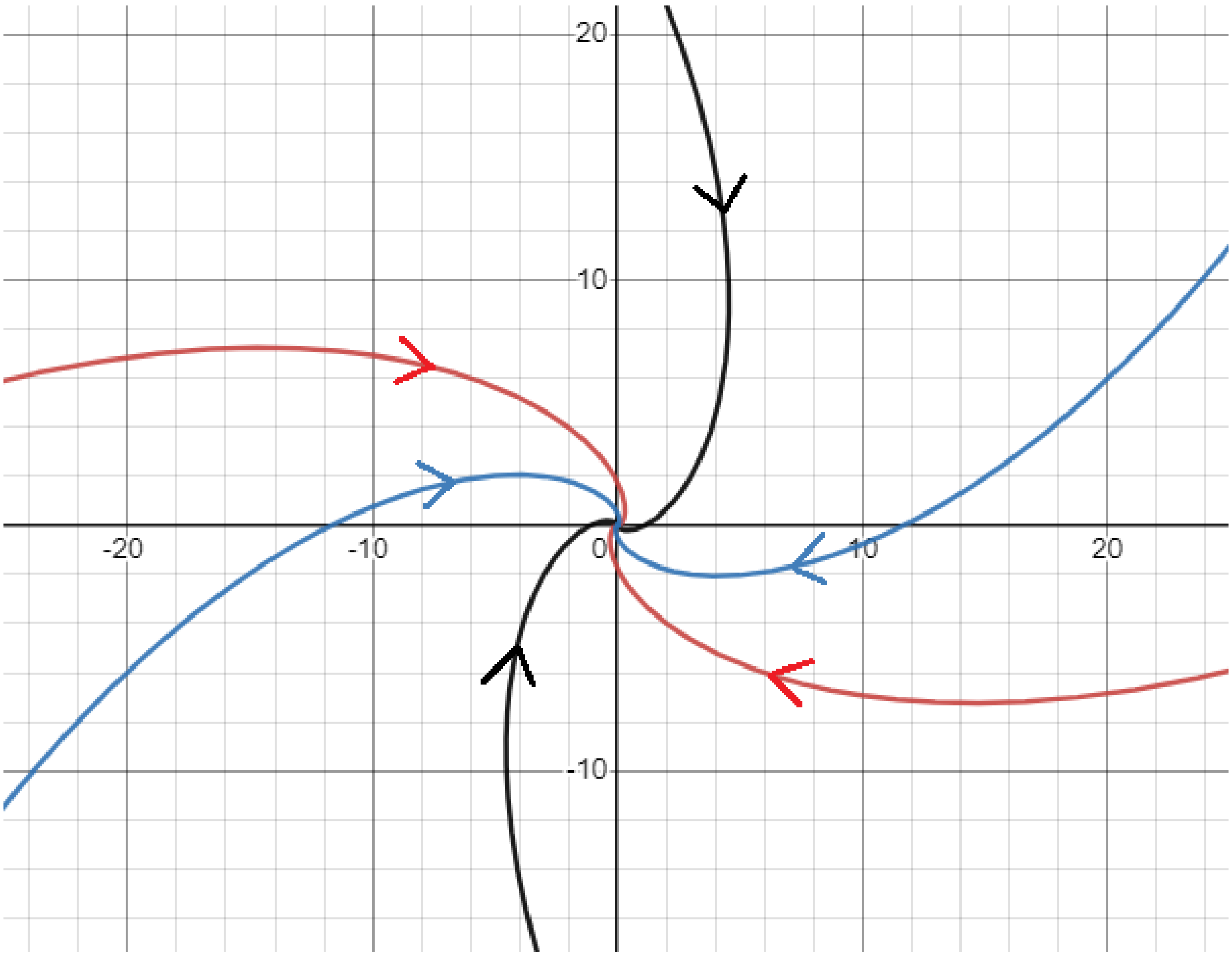
STEP 3: Solution

$$\begin{aligned}
e^{(-2+i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} &= \left(e^{-2t} e^{it}\right) \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \\
&= e^{-2t} \left(\cos(t) + i \sin(t)\right) \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \\
&= e^{-2t} \left(\cos(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \\
&\quad + i e^{-2t} \left(\cos(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$

STEP 4: Phase Portrait

Because of the e^{-2t} term solutions are spiraling inwards



8.

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} e^t \\ 2e^{2t} \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} \left|A - \lambda I\right| = & \begin{vmatrix} -1-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} \\ = & (-1-\lambda)(2-\lambda) - (2)(-1) \\ = & -2 + \lambda - 2\lambda + \lambda^2 + 2 \\ = & \lambda^2 - \lambda \\ = & \lambda(\lambda - 1) = 0 \end{aligned}$$

Which gives $\lambda = 0$ or $\lambda = 1$

STEP 2: $\lambda = 0$

$$\text{Nul } (A - 0I) = \left[\begin{array}{cc|c} -1-0 & 2 & 0 \\ -1 & 2-0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $-x + 2y = 0 \Rightarrow x = 2y$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

STEP 3: $\lambda = 1$

$$\text{Nul } (A - 1I) = \left[\begin{array}{cc|c} -1-1 & 2 & 0 \\ -1 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $-x + y = 0 \Rightarrow y = x$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

STEP 4: Homogeneous Solution

$$\mathbf{x_0}(t) = C_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

STEP 5: Variation of Parameters

$$\mathbf{x_p}(t) = u(t)\begin{bmatrix}2\\1\end{bmatrix} + v(t)\begin{bmatrix}e^t\\e^t\end{bmatrix}$$

$$\begin{bmatrix}2 & e^t\\1 & e^t\end{bmatrix}\begin{bmatrix}u'(t)\\v'(t)\end{bmatrix} = \begin{bmatrix}e^t\\2e^{2t}\end{bmatrix}$$

Denominator:

$$\begin{vmatrix}2 & e^t\\1 & e^t\end{vmatrix} = 2e^t - e^t = e^t$$

$$u'(t) = \frac{\begin{vmatrix}e^t & e^t\\2e^{2t} & e^t\end{vmatrix}}{e^t} = \frac{e^{2t} - 2e^{3t}}{e^t} = e^t - 2e^{2t}$$

$$u(t) = \int e^t - 2e^{2t}dt = e^t - e^{2t}$$

$$v'(t) = \frac{\begin{vmatrix}2 & e^t\\1 & 2e^{2t}\end{vmatrix}}{e^t} = \frac{4e^{2t} - e^t}{e^t} = 4e^t - 1$$

$$v(t) = \int 4e^t - 1dt = 4e^t - t$$

$$\mathbf{x_p}(t) = \left(e^t - e^{2t}\right)\begin{bmatrix}2\\1\end{bmatrix} + \left(4e^t - t\right)\begin{bmatrix}e^t\\e^t\end{bmatrix}$$

STEP 6: Answer

$$\begin{aligned}\mathbf{x}(t) =& \mathbf{x_0}(t) + \mathbf{x_p}(t) = C_1\begin{bmatrix}2\\1\end{bmatrix} + C_2e^t\begin{bmatrix}1\\1\end{bmatrix} + \left(e^t - e^{2t}\right)\begin{bmatrix}2\\1\end{bmatrix} + \left(4e^t - t\right)\begin{bmatrix}e^t\\e^t\end{bmatrix}\\ =& C_1\begin{bmatrix}2\\1\end{bmatrix} + C_2e^t\begin{bmatrix}1\\1\end{bmatrix} + e^t\begin{bmatrix}2\\1\end{bmatrix} - te^t\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\begin{bmatrix}2\\3\end{bmatrix}\end{aligned}$$

(This last simplification is optional)

9.

$$\begin{cases} x' = x - y^2 \\ y' = y - x^2 \end{cases}$$

STEP 1: Equilibrium Points

$$\begin{cases} x' = x - y^2 = 0 \\ y' = y - x^2 = 0 \end{cases}$$

The first equation gives $x = y^2$ and plugging this into the second

$$\begin{aligned} y &= x^2 \\ y &= \left(y^2\right)^2 \\ y &= y^4 \\ y^4 - y &= 0 \\ y\left(y^3 - 1\right) &= 0 \\ y &= 0 \text{ or } y^3 = 1 \\ y &= 0 \text{ or } y = 1 \end{aligned}$$

If $y = 0$ then $x = y^2 = 0^2 = 0$ which gives $(0, 0)$

If $y = 1$ then $x = y^2 = 1^1 = 1$ which gives $(1, 1)$

Equilibrium points: $(0, 0)$ and $(1, 1)$

STEP 2: Classification

$$\nabla F(x, y) = \begin{bmatrix} \frac{\partial(x-y^2)}{\partial x} & \frac{\partial(x-y^2)}{\partial y} \\ \frac{\partial(y-x^2)}{\partial x} & \frac{\partial(y-x^2)}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & -2y \\ -2x & 1 \end{bmatrix}$$

Case 1: $(0, 0)$

$$\nabla F(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is diagonal with eigenvalues $\lambda = 1 > 0$ and $\lambda = 1 > 0$ and so $(0, 0)$ is **unstable**

Case 2: $(1,1)$

$$\nabla F(1,1) = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = & \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \\ = & (1-\lambda)^2 - (-2)^2 \\ = & (\lambda - 1)^2 - 4 = 0 \end{aligned}$$

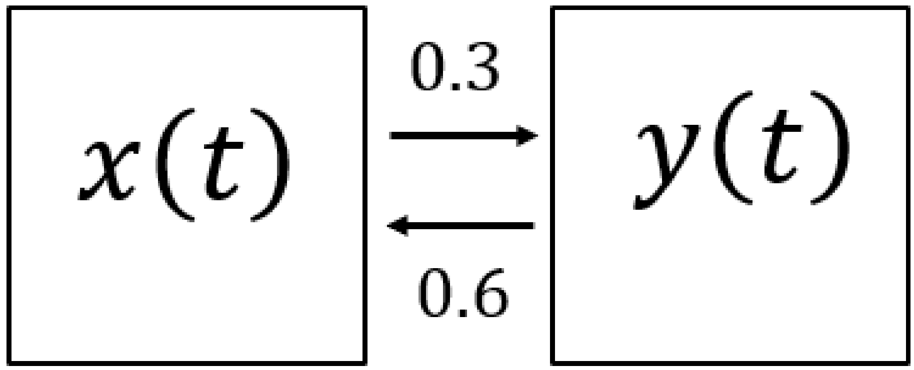
$(\lambda - 1)^2 = 4 \Rightarrow \lambda - 1 = -2$ or $\lambda - 1 = 2 \Rightarrow \lambda = -1$ or $\lambda = 3$

Since $\lambda = 2 > 0$ and $\lambda = -1 < 0$, $(1,1)$ is **a saddle**

STEP 3: Answer:

$(0,0)$ unstable
$(1,1)$ saddle

10. (a)



$$x'(t) = \text{In} - \text{Out} = (0.6)y(t) - (0.3)x(t) = -(0.3)x(t) + (0.6)y(t)$$
$$y'(t) = \text{In} - \text{Out} = (0.3)x(t) - (0.6)y(t)$$

$$A = \begin{bmatrix} -0.3 & 0.6 \\ 0.3 & -0.6 \end{bmatrix}$$

(b) **STEP 1:** In this case the system $\mathbf{x}' = A\mathbf{x}$ is

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -0.3 & 0.6 \\ 0.3 & -0.6 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -(0.3)x(t) + (0.6)y(t) \\ (0.3)x(t) - (0.6)y(t) \end{bmatrix}$$

$$\begin{cases} x' = -0.3x + 0.6y \\ y' = 0.3x - 0.6y \end{cases}$$

STEP 2: Setting $x'(t) = 0$ and $y'(t) = 0$ we get

$$\begin{cases} -0.3x + 0.6y = 0 \\ 0.3x - 0.6y = 0 \end{cases} \Rightarrow \begin{cases} -x + 2y = 0 \\ x - 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 2y \\ x = 2y \end{cases}$$

Therefore there are *infinitely* many equilibrium points, each of the form $(x, y) = (2y, y)$ where y is arbitrary

STEP 3: Stability

$$\nabla F(x, y) = \begin{bmatrix} \frac{\partial(-0.3x+0.6y)}{\partial x} & \frac{\partial(-0.3x+0.6y)}{\partial y} \\ \frac{\partial(0.3x-0.6y)}{\partial x} & \frac{\partial(0.3x-0.6y)}{\partial y} \end{bmatrix} = \begin{bmatrix} -0.3 & 0.6 \\ 0.3 & -0.6 \end{bmatrix}$$

$$\nabla F(2y, y) = \begin{bmatrix} -0.3 & 0.6 \\ 0.3 & -0.6 \end{bmatrix} = A$$

Eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -0.3 - \lambda & 0.6 \\ 0.3 & -0.6 - \lambda \end{vmatrix} \\
&= (-0.3 - \lambda)(-0.6 - \lambda) - (0.6)(0.3) \\
&= \cancel{(-0.3)(-0.6)} + 0.3\lambda + 0.6\lambda + \lambda^2 - \cancel{(0.6)(0.3)} \\
&= \lambda^2 + 0.9\lambda \\
&= \lambda(\lambda + 0.9) \\
&= 0
\end{aligned}$$

Which gives $\lambda = 0$ and $\lambda = -0.9 < 0$

In this case the equilibrium points $(2y, y)$ are **stable** but we also accept the answer **neither stable/unstable/saddle**