Homework 1 Solutions

1. (a) If $y = e^{rt}$, then we have $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. We can see which values of r result in a solution by substituting these expressions for y, y', and y'' into our differential equation:

$$y'' + 3y' - 10y = 0$$
$$\implies r^2 e^{rt} + 3re^{rt} - 10e^{rt} = 0$$
$$\implies (r^2 + 3r - 10)e^{rt} = 0$$

Because e^{rt} can never be zero, we see that $y = e^{rt}$ is a solution to the differential equation exactly when r is one of the roots of $r^2 + 3r - 10$, namely $r = 2$ or $r = -5$.

(b) If $y = t^4$, then $y' = 4t^3$ and $y'' = 12t^2$. Substituting these expressions into the differential equation, we get:

$$t^3 y'' + ty' - 8y = 0$$
$$\implies (t^3)(12t^2) + t(4t^3) - 8(t^4) = 0$$
$$\implies 12t^5 + 4t^4 - 8t^4 = 0$$
$$\implies 4t^4(3t - 1) = 0$$

This equation is only true for $t = 0$ and $t = 1/3$; for any other value of t it is false. Therefore, $y = t^4$ is not a solution.

2. (a) 2nd order because the highest derivative of y which appears is y'', linear, inhomogenous because of the $13t^3$ term.

(b) 1st order because the highest derivative of y which appears is y', nonlinear because of the $(y')^2$ term.

(c) 3rd order because the highest derivative of y which appears is y''', linear, homogenous because the right side of the equation is 0.

(d) 1st order because the highest derivative of y which appears is y', linear, homogenous because the right side of the equation is 0.

(e) 1st order because the highest derivative of y which appears is y', nonlinear because of the y^2 term.

3.
$$\frac{dy}{dt} = 2y + 3$$
$$\implies \left(\frac{1}{2y+3}\right)\frac{dy}{dt} = 1$$
$$\implies \frac{1}{2}\left(\ln|2y+3|\right)' = 1$$
$$\implies \ln|2y+3| = 2t + C$$
$$\implies e^{\ln|2y+3|} = |2y+3| = e^{2t+C} \qquad \text{(by exponentiating both sides)}$$
$$\implies |2y+3| = e^{C}e^{2t}.$$

e^C is just another constant, so we can rewrite it as just C:

$$|2y+3| = Ce^{2t}$$
$$\implies 2y + 3 = \pm Ce^{2t}.$$

Again, because C is arbitrary we can rewrite $\pm C$ as simply C, so we end up with
$$y(t) = \frac{Ce^{2t} - 3}{2}.$$

By plugging the initial condition $y(0) = 3$ into this equation, we get

$$\frac{Ce^0 - 3}{2} = \frac{C-3}{2} = 3$$
$$\implies C = 9.$$

Therefore,
$$y(t) = \frac{9e^{2t} - 3}{2} = 4.5e^{2t} - 1.5.$$

4. (a) If
$$y = Ae^{2t} + Bte^{2t},$$

then
$$y' = 2Ae^{2t} + 2Bte^{2t} + Be^{2t}$$

and
$$y'' = 4Ae^{2t} + 4Bte^{2t} + 4Be^{2t},$$

so
$$y'' - 4y' + 4y$$
$$= (4Ae^{2t}+4Bte^{2t}+4Be^{2t})-4(2Ae^{2t}+2Bte^{2t}+Be^{2t})+4(Ae^{2t}+Bte^{2t})$$
$$= 0.$$

(b) Following the hint, consider $(e^{-2t}y)''$, which we can also write as $\frac{d^2}{dt^2}(e^{-2t}y)$. Evaluating this, we see that

$$\frac{d^2}{dt^2}(e^{-2t}y)$$

$$\begin{aligned}
&= \frac{d}{dt}(y'e^{-2t} - 2ye^{2t}) \\
&= y''e^{-2t} - 2y'e^{-2t} + 4ye^{-2t} - 2e^{-2t}y' \\
&= y''(e^{-2t}) - 4y'(e^{-2t}) + 4y(e^{-2t}) \\
&= e^{-2t}(y'' - 4y' + y). \qquad (1)
\end{aligned}$$

Because we are assuming that y solves the ODE $y'' - 4y' + 4y = 0$, the expression (1) must be equal to 0. Therefore,

$$\frac{d^2}{dt^2}(e^{-2t}y) = 0,$$

so $e^{-2t}y$ must be of the form $A + Bt$ because it vanishes after two derivatives. Therefore, y must be of the form $Ae^{2t} + Bte^{2t}$.

5. Let $B(t)$ denote the number of bacteria at time t. Then the fact that the rate of change is equal to three times the number of bacteria can be expressed as the following ODE:

$$\frac{dB}{dt} = 3B.$$

We can solve this with separation of variables:

$$\int \frac{1}{B} dB = \int 3dt$$

$$\implies \ln B = 3t + C$$

$$\implies B(t) = Ce^{3t} \qquad \text{(by exponentiating)}$$

If we initially have a population of size B_0, i.e. $B(0) = B_0$, then

$$B_0 = Ce^0 = C$$

$$\implies B(t) = B_0 e^{3t}.$$

To find the amount of time it takes for the population to double, we want to find t such that

$$B_0 e^{3t} = 2B_0.$$

By dividing both sides of this equation by B_0, we have

$$e^{3t} = 2.$$

We can already see that this value of t does not depend on the initial population size B_0 because B_0 no longer appears in the equation. By taking the logarithm of both sides, we see that

$$3t = \ln(2)$$

$$\implies t = \frac{\ln(2)}{3}.$$

Therefore, the size of the population will double at time $t = \ln(2)/3$, regardless of the initial population size.