APMA 0359 - Homework II Solutions

September 17, 2023

1. Do the following ODE satisfy the assumptions of the Existence-Uniqueness Theorem from Lecture? Why or why not?

   Recall:

   Theorem 0.1 (Existence/Uniqueness). *Consider the ODE*

   $$\begin{cases} y' = f(y,t) \\ y(0) = y_0 \end{cases}$$

   where y_0 is a given number. If f and $\frac{\partial f}{\partial y}$ are continuous, then the ODE has a unique solution $y = y(t)$ for t close enough to 0.

   (a)
   $$\begin{cases} \frac{dy}{dt} = \frac{y}{t^2+1} \\ y(-1) = 9 \end{cases}$$

   Solution: We have that

   $$f(y,t) = \frac{y}{t^2+1}$$

   and

   $$\frac{\partial}{\partial y}(f(y,t)) = \frac{\partial}{\partial y}\left(\frac{y}{t^2+1}\right) = \frac{1}{t^2+1}.$$

   Both are continuous everywhere, so the theorem holds.

   (b)
   $$\begin{cases} \frac{dy}{dt} = y^2(|t| + y) \; y(0) = 2 \end{cases}$$

   Solution: We have that

   $$f(t,y) = y^2(|t| + y) = y^2|t| + y^3$$

   and

   $$\frac{\partial}{\partial y}(f(t,y)) = 2y|t| = y^2|t| + 2y^2.$$

   Both are continuous everywhere, so the theorem holds.

(c)

$$\begin{cases} \frac{dy}{dt} = \frac{1}{y+1} \\ y(1) = -1 \end{cases}$$

Solution:
We have that

$$f(y,t) = \frac{1}{y+1} = (y+1)^{-1}$$

and

$$\frac{\partial}{\partial y}(f(y,t)) = \frac{\partial}{\partial y}((y+1)^{-1}) = -(y-1)^{-2}.$$

This ODE does not satisfy the assumptions of the theorem because $f(y,t)$ is not continuous near $t = 1$. Using the initial condition, we see that $y(t = 1) = -1$ and $f(-1,1) = \frac{1}{(-1)+1}$ which is undefined $\left(\frac{1}{0}\right)$.

2. Consider the equation

$$y' = (y+2)(y-1)(y-3).$$

(a) Find the equilibrium solutions

Solution: We set $f(y) = 0$ and find the equilibrium solutions $y = -2, y = 1$, and $y = 3$.

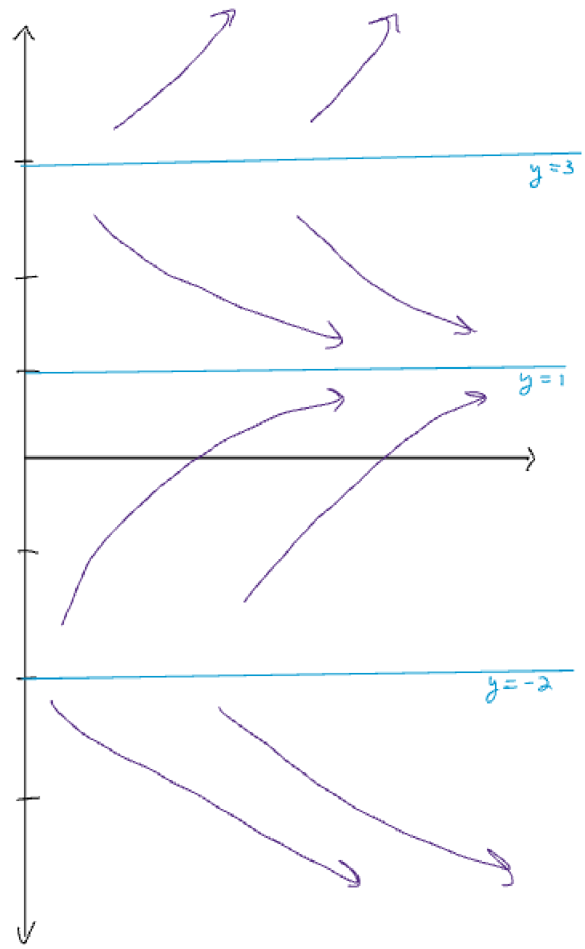
(b) Draw a bifurcation diagram (that table with signs)

Solution:

y	$-\infty$	-2		1		3	∞
$y+2$	$-$	0	$+$		$+$		$+$
$y-1$	$-$		$-$	0	$+$		$+$
$y-3$	$-$		$-$		$-$	0	$+$
y'	$-$	0	$+$	0	$-$	0	$+$
y	↘		↗		↘		↗

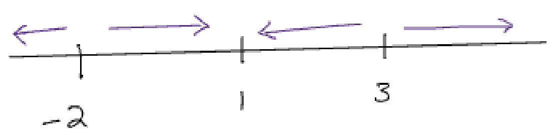
(c) Draw a sketch of the equilibrium solutions and a couple of solutions between them (that graph with the curves).

Solution:



(d) Classify the equilibrium solutions as stable/unstable/bistable.

Solution: We see that $y = 1$ is a stable equilibrium, and $y = -2$ and $y = 3$ are unstable equilibria.



3. Solve using separation of variables.

(a) (Implicit form)

$$\frac{dy}{dt} = \frac{t\sec(y)}{y(e^{t^2})}$$

Solution: First, we separate variables to find

$$\frac{y}{\sec(y)}dy = te^{-t^2}dt.$$

Using the trig identity $1/\sec(y) = \cos(y)$ and integrating both sides, we obtain

$$\int y\cos(y)dy = \int te^{-t^2}dt.$$

Using integration by parts on the LHS ($u = y$ and $dv = \cos(y)$) and u-substitution on the RHS ($u = -t^2$), we arrive at

$$y\sin(y) + \cos(y) = \frac{-e^{-t^2}}{2} + C.$$

(b) (Remember the inital condition)

$$\begin{cases} \frac{dy}{dt} = \frac{2t+\sec^2(t)}{2y} \\ y(0) = -5 \end{cases}$$

Solution: Separating and integrating both sides, we find

$$\int 2ydy = \int t + \sec^2(t))dt.$$

Evaluating the integrals, we obtain

$$\begin{aligned} y^2 &= t^2 + \tan(t) + C \\ \implies y &= \pm\sqrt{t^2 + \tan(t) + C}. \end{aligned}$$

Applying the intial condition $y(0) = -5$, we see that we must use $-(\sqrt{t^2 + \tan(t) + C})$ and find

$$\begin{aligned} -5 &= -\sqrt{0 + C} \\ 5 &= \sqrt{C} \\ \implies C &= 25. \end{aligned}$$

Thus, we arrive at the final answer,

$$y(t) = -\sqrt{t^2 + \tan(t) + 25}.$$

(c) (Beware of hidden solutions, see last example in the notes

$$\frac{dy}{dt} = \frac{-t(y-1)^2}{3}$$

Solution: First, we consider if $y \neq 1$. Then, we can separate to find

$$\frac{1}{(y-1)^2}dy = \frac{-t}{3}dt.$$

Integrating both sides, we find

$$\int (y-1)^{-2}dy = \int \frac{-t}{3}dt.$$

Evaluating, we obtain

$$\frac{-1}{y-1} = \frac{-t^2}{6} + c.$$

We still need to solve for y (-1 point if student stopped here). Taking the reciprocal of both sides, we find

$$\begin{aligned}
-(y-1) &= \frac{1}{\frac{-t^2}{6} + c} \\
&= \frac{6}{-t^2 + 6c} \\
\implies -y &= \frac{6}{-t^2 + 6c} - 1 \\
\implies y(t) &= 1 + \frac{6}{t^2 - 6c}.
\end{aligned}$$

If $y = 1$, we also have a solution. The above case only worked for $y \neq 1$. Thus, we miss a solution as explained in class if only the above is worked out.

4. Suppose you're trying to model a population of bunnies. Initially there are 100 rabbits, and you notice that the initial growth rate is 0.08 bunnies/day. Moreover, due to limiting resources, it seems that the bunny population doesn't exceed 1000 rabbits.

   (a) Set up a differential equations model for the number of rabbits and follow the steps from lecture to solve it. You can skip the partial fractions part if you wish.

Solution: First, we recognize set up the differential equation as in lecture as

$$\begin{cases} \frac{dy}{dt} = 0.08y(1 - \frac{y}{1000} \\ y(0) = 100. \end{cases}$$

Separating and rewriting the LHS, we get

$$\frac{1000}{y(1000-y)} dy = 0.08dt$$

Applying partial fraction decomposition on the LHS, we rewrite and integrate as

$$\int \frac{1}{y} - \frac{1}{1000-y} dy = \int 0.08dt.$$

Evaluating and simplifying, we obtain

$$\begin{aligned}
\ln|y| - \ln|1000-y| &= 0.08t + C_1 \\
\ln\left|\frac{y}{1000-y}\right| &= 0.08t + C_1 \\
\implies \left|\frac{y}{1000-y}\right| &= e^{0.08t + C_1}.
\end{aligned}$$

Next, we aim to solve for y. Simplifying, we find

$$
\begin{aligned}
\frac{y}{1000-y} &= \pm e^{C_1} e^{0.08t} \\
&= C_2 e^{0.08t}
\end{aligned}
$$

Cross multiplying, we get

$$
\begin{aligned}
y &= (1000-y)C_2 e^{0.08t} \\
&= 1000C_2 e^{0.08t} - yC_2 e^{0.08t}
\end{aligned}
$$

Lastly, we arrive at,

$$
\begin{aligned}
y(1 + C_2 e^{0.08t}) &= 1000C_2 e^{0.08t} \\
\implies y &= \frac{1000C_2 e^{0.08t}}{1 + C_2 e^{0.08t}}
\end{aligned}
$$

Using the I.C., we find,

$$
\begin{aligned}
100 &= \frac{1000(C_2)}{1 + C_2} \\
100 + 100C_2 &= 1000C_2 \\
100 &= 900C_2 \\
\implies C_2 &= \frac{1}{9}.
\end{aligned}
$$

Thus, we arrive at the solution (with optional simplification),

$$
\begin{aligned}
y =& \frac{1000e^{0.08t}}{9(1 + (\frac{1}{9}e^{0.08t}))} \\
=& \frac{1000e^{0.08t}}{9 + e^{0.08t}} \left(\frac{e^{-0.08t}}{e^{-0.08t}} \right) \\
=& \frac{1000}{9e^{-0.08t} + 1}.
\end{aligned}
$$

(b) What is the limit of the solution in (a) as $t \to \infty$?

Solution: We know that the

$$\lim_{t \to \infty} e^{-0.08t} = 0.$$

Thus, we find that

$$\lim_{t \to \infty} \frac{1000}{9e^{-0.08t} + 1} = \frac{1000}{0 + 1} = 1000.$$

(c) For which t do we have $y(t) = 900$? Please find an exact formula for t, as well as an approximate value (using your calculator).

Solution: We set up

$$900 = \frac{1000}{9e^{-0.08t} + 1}.$$

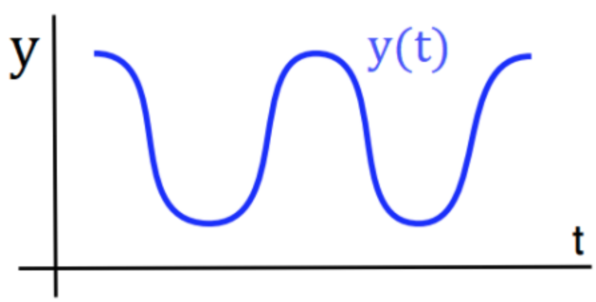
Simplifying, we obtain

$$e^{-0.08t} = \frac{1}{81}.$$

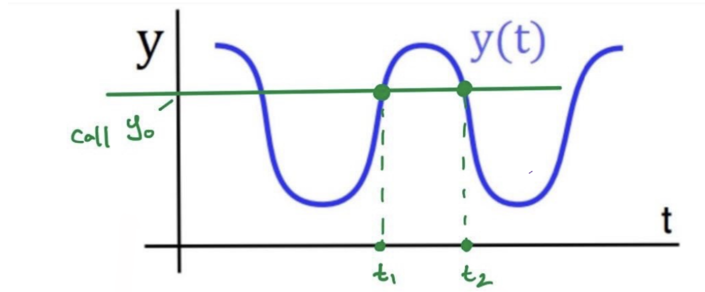
Taking the ln of both sides, we find

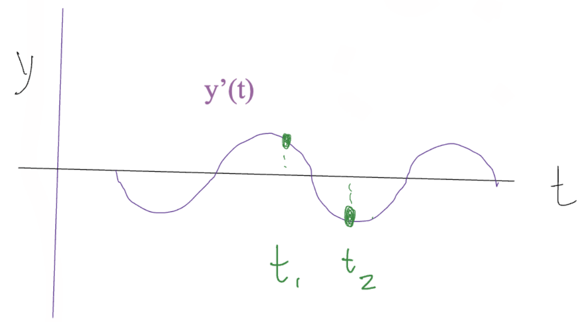
$$\ln e^{-0.08t} = \ln\left(\frac{1}{81}\right)$$
$$\implies t = \frac{-\ln\left(\frac{1}{81}\right)}{0.08} \approx 54.9306.$$

5. Show that the function $y(t)$ whose graph is below cannot be a solution of an ODE of the form $y'(t) = f(y(t))$.



Solution: We recall the definition of a function: A function from a set X to a set Y assigns to each element of X exactly one element of Y. Thus, the given form implies that $y(t)$ can be expressed as an input a of a function describing $y'(t)$. This form is violated if the same value of $y(t)$ produces more than one value of $y'(t)$. Thus, following the hint, we find a value of $y(t)$ which is the same for different t values and produce two different values of $y'(t)$. By inspection of the graph, we choose the following t_1 and t_2 and roughly sketch the corresponding plot for $y'(t)$.





Thus, we find that

$$y'(t_1) = f(y(t_1)) = f(y_0) > 0, \text{ and}$$
$$y'(t_2) = f(y(t_1)) = f(y_0) < 0.$$

Thus, we arrive at a contradiction. This means that $y'(t)$ cannot be expressed as a function of $y(t)$ and cannot be written in the form $y'(t) = f(y(t))$.