Homework 3 Solutions

Problem 1. Steps to solve using integrating factors:

1.) Write the equation in this form:

$$\frac{dy}{dx} + a(x)y = b(x) \tag{1}$$

2.) Solve for the integrating factor:

$$\mu = e^{\int a(x)dx}$$

3.) Multiply (1) by μ, and rewrite the left-hand side as a derivative:

$$\mu\left(\frac{dy}{dx} + a(x)y\right) = \mu b(x)$$

$$\implies \frac{d}{dx}(\mu y) = \mu b(x)$$

4.) Integrate both sides and use the Fundamental Theorem of Calculus

$$\int \frac{d}{dx}(\mu y)dx = \int \mu b(x)dx$$

$$\implies \mu y = B(x) + C$$

5.) Divide by μ, and you have the solution.

(a) Step 1:

$$\frac{dy}{dt} - 2y = t^2 e^{2t}$$

Step 2:

$$a(t) = -2$$

$$\implies \int a(t)dt = \int -2dt = -2t$$

$$\implies \mu = e^{\int a(t)dt} = e^{-2t}$$

Step 3:

$$e^{-2t}\left(\frac{dy}{dt} - 2y\right) = e^{-2t}\left(t^2 e^{2t}\right) = t^2$$

$$\Longrightarrow \frac{d}{dt}\left(e^{-2t}y\right) = t^2$$

Step 4:

$$\int \frac{d}{dt}\left(e^{-2t}y\right)dt = \int t^2 dt$$

$$\Longrightarrow e^{-2t}y = \frac{t^3}{3} + C$$

Step 5:

$$y = \left(\frac{t^3}{3} + C\right)e^{2t}$$

(b) Step 1:

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{\sin(t)}{t}$$

Step 2:

$$a(t) = \frac{2}{t}$$

$$\Longrightarrow \int a(t)dt = \int \frac{2}{t}dt = 2\ln(t)$$

$$\mu = e^{\int a(t)dt} = e^{2\ln(t)} = \left(e^{\ln(t)}\right)^2 = t^2$$

Step 3:

$$t^2\left(\frac{dy}{dt} + \frac{2}{t}y\right) = t^2\left(\frac{\sin(t)}{t}\right) = t\sin(t)$$

$$\Longrightarrow \frac{d}{dt}\left(t^2 y\right) = t\sin(t)$$

Step 4:

$$\int \frac{d}{dt}\left(t^2 y\right)dt = \int t\sin(t)dt$$

$$\Longrightarrow t^2 y = \sin(t) - t\cos(t) + C$$

(Using integration my parts on the left side)

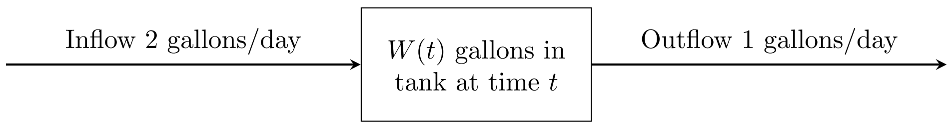
Step 5:

$$y = \frac{\sin(t) - t\cos(t) + C}{t^2}$$

Problem 2. (a) Compartment model for water:

$$t = \text{time measured in days}$$

$$W(t) = \text{amount of water in gallons in the tank at time } t$$



$$\frac{dW(t)}{dt} \text{ gallons/day} = 2 \text{ gallons/day} - 1 \text{ gallon/day} = 1 \text{ gallon/day}$$

$$\implies \frac{dW}{dt} = 1.$$

Therefore,

$$\int_0^t \frac{dW(s)}{ds} ds = \int_0^t 1 ds$$
$$\implies W(t) - W(0) = t$$
$$\implies W(t) = W(0) + t.$$

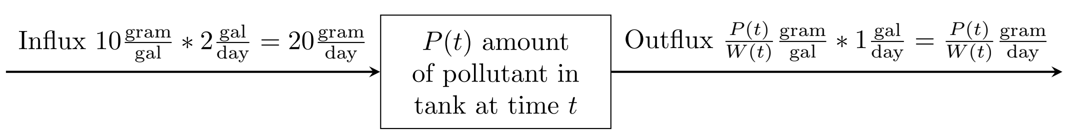
We know that $W(0) = 100$, so

$$W(t) = 100 + t.$$

Therefore, the amount of water reaches 200 gallons at $t = 100$.

(b) Compartment model for chemical pollutant:

$$P(t) = \text{amount of pollutant in grams in the tank at time t}$$



Therefore,

$$\frac{dP}{dt} = 20 - \frac{P}{W(t)} = 20 - \frac{P}{100+t}.$$

We can solve this ODE with integrating factors:

Step 1:

$$\frac{dP}{dt} + \frac{1}{100+t} P = 20$$

Step 2:

$$a(t) = \frac{1}{100+t}$$

$$\implies \mu = e^{\int a(t)dt} = e^{\ln(100+t)} = 100 + t$$

Step 3:

$$\mu\left(\frac{dP}{dt}+\frac{1}{100+t}P\right)=\frac{d}{dt}(\mu P)=20\mu=2000+20t$$

Step 4:

$$\mu P=\int(2000+20t)=2000t+10t^2+C$$

Step 5:

$$P(t)=\frac{2000t+10t^2+C}{\mu}=\frac{2000t+10t^2+C}{100+t}$$

We know that $P(0)=0$ because the tank contains only clean water at time 0. Plugging in this initial condition gives us $P(0)=C/100$, so $C=0$, and thus the solution is

$$P(t)=\frac{2000t+10t^2}{100+t}=10t*\frac{200+t}{100+t}.$$

(c) When $t=100$, the tank contains $P(100)$=1500 grams of pollutant.

Problem 3. (a) Time t is in independent variable, and its unit is months. Rabbit population $P(t)$ is a dependent variable, it depends on t, and its unit is rabbits.

(b) Using the trick from class,

$$\frac{P(t+h)-P(t)}{h}=0.1P(t)-0.3P(t)+30=-0.2P(t)+30$$

$$\implies\frac{dP}{dt}=-0.2P(t)+30,$$

and the initial condition is

$$P(0)=50$$

where $t=0$ is January.

(c) Equilibrium:

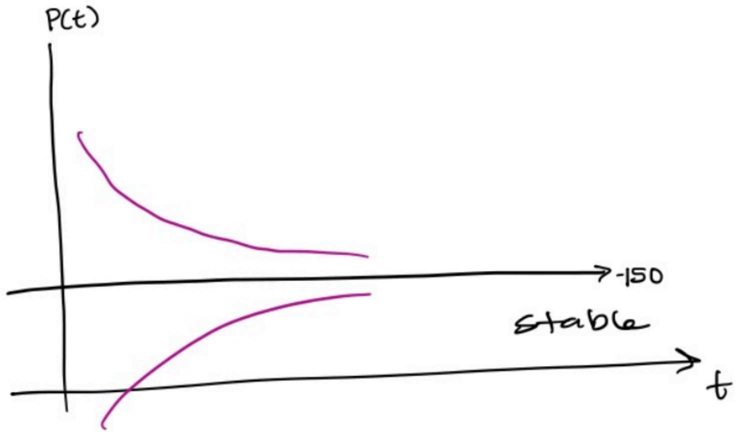
$$0=-0.2P(t)+30$$

$$\implies P(t)=150$$

Bifurcation diagram:

	$-\infty$	150	∞
P'	$+$	0	$-$
P	$\nearrow$	$\longrightarrow$	$\searrow$

Plot of equilibrium solution and two other solutions:



The equilibrium is stable.

(d) We can solve the ODE with integrating factors:

Step 1:

$$\frac{dP(t)}{dt} + 0.2P(t) = 30$$

Step 2:

$$\mu = e^{\int 0.2dt} = e^{0.2t}$$

Step 3:

$$e^{0.2t}\left(\frac{dP(t)}{dt} + 0.2P(t)\right) = 30e^{0.2t}$$

$$\implies \frac{d}{dt}\left(e^{0.2t}P(t)\right) = 30e^{0.2t}$$

Step 4:

$$\int \frac{d}{dt}\left(e^{0.2t}P(t)\right)dt = \int 30e^{0.2t}dt$$

$$\implies e^{0.2t}P(t) = 150e^{0.2t} + C$$

Step 5:

$$P(t) = 150 + Ce^{-0.2t}$$

$$P(0) = 150 + C = 50$$

$$\implies C = -100$$

$$\implies P(t) = 150 - 100e^{-0.2t}$$

(e) Since $e^{-0.2t} \to 0$ as $t \to \infty$, we see that

$$\begin{aligned}\lim_{t\to\infty P(t)} &= \lim_{t\to\infty}(150 - 100e^{-0.2t}) \\ &= \lim_{t\to\infty}(150) + \lim_{t\to\infty}(-100e^{-0.2t}) \\ &= 150,\end{aligned}$$

so the population of bunnies will approach 150 as $t \to \infty$.

Problem 4. If we let T denote the temperature of the pie, then we can set up the following ODE:

$$\frac{dT}{dt} = k(22 - T) = -kT + 22k,$$

where k is unknown for now. We can solve this with integrating factors:

$$\begin{aligned}&\frac{dT}{dt} + kT = 22k \\ \implies e^{kt}\left(\frac{dT}{dt} + kT\right) &= \frac{d}{dt}\left(e^{kt}T\right) = 22ke^{kt} \\ &\implies e^{kt}T = 22e^{kt} + C \\ &\implies T(t) = 22 + Ce^{-kt}.\end{aligned}$$

We know that $T(0) = 40$, and plugging this in gives $40 = 22 + C$, so $C = 18$. Our solution is now

$$T(t) = 22 + 18e^{-kt}.$$

We also know that $T(1) = 35$, and plugging this in gives

$$\begin{aligned}35 &= 18e^{-k} + 22 \\ \implies e^{-k} &= \frac{13}{18} \\ \implies k &= -\ln\left(\frac{13}{18}\right) \approx 0.33,\end{aligned}$$

so our solution is now

$$T(t) = 18e^{-0.33t} + 22.$$

We need to find t such that $T(t) = 25$:

$$\begin{aligned}18e^{-0.33t} + 22 &= 25 \\ \implies e^{-0.33t} &= \frac{1}{6} \\ \implies t &= \frac{\ln(1/6)}{-0.33} \approx 5.43 \text{ minutes}\end{aligned}$$