APMA 0359 - Homework 4 Solutions

October 4, 2023

1. Apply Euler's method by hand with $N = 4$ to find y_0, y_1, y_2, y_3, y_4 on $[0, 1]$ where

$$\begin{cases} y' = -2y + 3t \\ y(0) = 1 \end{cases}$$

Solution:

Step 1: Find step size $h = \frac{a-bN}{=} \frac{1}{4}$

Step 2: Find $y_0, .., y_n$ with $y_n = y_{n-1} + h(f(y_{n-1}, t_{n-1},$ where $t_n = hn$.

n	t_n	Method for y_n	y_n	$f(y_n, t_n) = -2y_n + 3t_n$
0	0	$y_0 = y(0) = 1$	1	$-2(1) + 3(0) = -2$
1	$\frac{1}{4}$	$y_1 = (1) + \frac{1}{4}(-2) = \frac{1}{2}$	$\frac{1}{2}$	$-2(\frac{1}{2}) + 3(\frac{1}{4}) = -\frac{1}{4}$
2	$\frac{1}{2}$	$y_2 = (\frac{1}{2}) + \frac{1}{4}(-\frac{1}{4}) = \frac{7}{16}$	$\frac{7}{16}$	$-2(\frac{7}{16}) + 3(\frac{1}{2})0 = -\frac{1}{4}$
3	$\frac{3}{4}$	$y_3 = (\frac{7}{16}) + \frac{1}{4}(\frac{5}{8}) = \frac{19}{32}$	$\frac{19}{32}$	$-2(\frac{19}{32}) + 3(\frac{3}{4}) = \frac{17}{16}$
4	1	$y_4 = \frac{19}{32} + \frac{1}{4}(\frac{17}{16}) = \frac{55}{64}$	$\frac{55}{64}$	

2. Solve the following exact ODE. Leave your answer in implicit form. Don't forget to check for exactness.

(a)

$$\frac{dy}{dt} - \left(\frac{e^x \sin(y) - 2y\sin(x)}{e^x \cos(y) + 2\cos(x)} \right)$$

Solution: We rewrite in the proper form as

$$\underbrace{[e^x \cos(y) + 2\cos(x)]}_{Q} dx + \underbrace{[e^x \sin(y) - 2y\sin(x)]}_{P} dx$$

We check for exactness by taking the following derivatives:

$$Q_x = (e^x \cos(y) + 2\cos(x))_x = e^x \cos(y) - 2\sin(x))$$
$$P_y = (e^x \sin(y) - 2y\sin(x))_y = -2\sin(x) + e^x \cos(y).$$

Next, we find f. We see that

$$f_x = P \implies \int e^x \sin(y) - 2y\sin(x)dx = e^x \sin(y) + 2y\cos(x) + g(y)$$

and

$$f_y = Q \implies \int (e^x \cos(y) + 2\cos(x))dy = e^x \sin(y) + h(x).$$

This implies that

$$f(x, y) = e^x \sin(x) + 2y\cos(x).$$

Thus, we arrive at the solution,

$$e^x \sin(x) + 2y\cos(x) = C.$$

(b)

$$\frac{dy}{dx} = -\left(\frac{f(x)}{g(y)}\right).$$

Solution: We first rewrite as

$$\underbrace{g(y)}_{Q}\, dy + \underbrace{f(x)}_{P}\, dx = 0.$$

For exactness, we check that

$$Q_x = [g(y)]_x = 0 = [f(x)]_y = P_y.$$

Thus, we find

$$f_x = P \implies F = \int f(x)dx$$

and

$$f_y = Q \implies G = \int g(y)dy.$$

Thus, we have that $F + G = C$.

3. Find the general solution of the following ODE:

(a) y " = y' + y

Solution: We first rewrite this as

$$y'' - y' - y = 0$$

which gives us

$$r^2 - r - 1 = 0.$$

We fund the roots using the quadratic formula as $r = \frac{1\pm\sqrt{5}}{2}$. Thus,

$$y = Ae^{\frac{1+\sqrt{5}}{2}t} + Be^{\frac{1-\sqrt{5}}{2}t}.$$

(b) 6y" - 7y' + 2y = 0

Solution: We rewrite as

$$y'' - \frac{7}{6}y' + \frac{1}{3}y = 0.$$

Thus, we have

$$r^2 - \frac{7}{6}r + \frac{1}{3} = 0.$$

Using the quadratic formula, we find $r = \frac{\frac{7}{6} \pm \frac{1}{6}}{2}$. Thus, we have

$$y = Ae^{\frac{2}{3}t} + Be^{\frac{1}{2}t}.$$

(c) An ODE whose auxilary equation is

$$(r-1)r(r+1)(r+2) = 0.$$

Solution: We easily obtain the roots $r = 1, 0, -1, -2$. Thus,

$$y = A + Be^t + Ce^{-t} + De^{-2t}.$$

4. Solve the following ODE

$$\begin{cases} y'' - 3y' - 28y = 0 \\ y(0) = 3 \\ y'(0) - 1. \end{cases}$$

Solution: We have that $r^2 - 3r - 28 = 0$. Thus, we have

$$(r-7)(r+4) = 0 \implies r = 7, -4.$$

Thus, we have

$$y = Ae^{7t} + Be^{-4t}.$$

To use both initial conditions, we must also find y'. Thus, we have that

$$y = 7Ae^{7t} - 4Be^{-4t}.$$

Using $y(0) = 3$ and $y'(0) = 1$, we find

$$y(0) = A + B = 3 \implies A = 3 - B,$$

and

$$y'(0) = 7A - 4B = 7(3-B) - 4B = -1 \implies B = 2.$$

Thus, we have that $A = 1$ and $B = 2$. Our solution is written as

$$y = e^{7t} + 2e^{-4t}.$$