LECTURE: BOUNDARY-VALUE PROBLEMS

1. Motivation

Here is some motivation in what's to follow

Example 1:

Consider $y'' - 5y' + 6y = 0$

Aux: $r^2 - 5r + 6 = 0 \Rightarrow r = 2$ or $r = 3$

In the example below, the roots 2 and 3 are called the eigenvalues:

Eigenvalues: $\lambda = 2$ and $\lambda = 3$

The general solution is

$$y = Ae^{2t} + Be^{3t}$$

Notice: Starting from e^{2t} and e^{3t} and using only addition and multiplication, we can get the general solution $y = Ae^{2t} + Be^{3t}$

In the example below, the "building blocks" e^{2t} and e^{3t} are called the eigenfunctions:

Eigenfunctions: $y = e^{2t}$ and $y = e^{3t}$

We "ignore" the constants because for us, there is no difference between the building blocks e^{2t} and e^{3t} and the solution $y = Ae^{2t} + Be^{3t}$ we can easily go from one to the other. In Linear Algebra, this is called a "basis" of solutions.

2. Boundary-Value Problems (continued)

Video: Boundary-Value Problems

Example 2: (continued)

Find the values of λ for which the ODE has nonzero solutions

$$\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

Auxiliary Equation: $r^2 = \lambda$

Cases 1 + 2: $\lambda > 0$ and $\lambda = 0$

No nonzero solutions

Case 3: $\lambda < 0$

In this case $\lambda = -\omega^2$ where $\omega > 0$

Ex: $\lambda = -9 = -3^2$ so $\omega = 3$

Aux: $r^2 = \lambda = -\omega^2 \Rightarrow r = \pm\omega i$

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$y(0) = A\cos(0) + B\sin(0) = A = 0$$

$$y(t) = B\sin(\omega t)$$

$$\begin{aligned} y(\pi) &= 0 \\ \cancel{B}\sin(\omega\pi) &= 0 \qquad B \neq 0 \text{ because otherwise } y = 0\sin(\omega t) = 0 \\ \sin(\omega\pi) &= 0 \\ \omega\pi &= \pi m \\ \omega &= m \end{aligned}$$

Here m can be any integer, but since $\omega > 0$, m needs to be any **positive** integer, $m = 1, 2, 3, \cdots$

Then $\lambda = -\omega^2 = -m^2 = -1, -4, -9, \cdots$

Moreover, from the above, we get $y = B\sin(\omega t) = B\sin(mt)$

Epic Conclusion:

Eigenvalues: $\lambda = -m^2\ (m = 1, 2, \cdots) = -1, -4, -9, \cdots$

Eigenfunctions:

$$y = \sin(mt)\ (m = 1, 2, \cdots) = \sin(t), \sin(2t), \sin(3t), \cdots$$

Note: Compare this from before where we had $\lambda = 2, 3$ and $y = e^{2t}, e^{3t}$ it's an infinite version of the previous example

Note: This is called eigenvalue/eigenfunction by comparison with $A\mathbf{v} = \lambda\mathbf{v}$ from linear algebra. Here we have $y'' = \lambda y$

3. Derivative Example

Video: Derivative Example

Example 3:

Find the eigenvalues and eigenfunctions of

$$\begin{cases} y'' = \lambda y \\ y'(0) = 0 \\ y'(1) = 0 \end{cases}$$

Auxiliary Equation: $r^2 = \lambda$

Case 1: $\lambda > 0$

Then $r^2 = \lambda = \omega^2 \Rightarrow r = \pm\omega$, in which case we get

$$y(t) = Ae^{\omega t} + Be^{-\omega t}$$

$$y'(t) = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

$$y'(0) = A\omega - B\omega = 0 \Rightarrow A\cancel{\omega} = B\cancel{\omega} \Rightarrow A = B$$

$$y(t) = Ae^{\omega t} + Ae^{-\omega t}$$

$$\begin{aligned} y'(1) &= 0 \\ A\omega e^{\omega} - A\omega e^{-\omega} &= 0 \\ \cancel{A\omega} e^{\omega} &= \cancel{A\omega} e^{-\omega} \\ e^{\omega} &= e^{-\omega} \\ \omega &= -\omega \\ \omega &= 0 \end{aligned}$$

But then $\lambda = \omega^2 = 0^2 = 0$ which contradicts $\lambda > 0 \Rightarrow\Leftarrow$

Conclusion: In this case, we have no nonzero solutions

Case 2: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (repeated twice)

$$y(t) = A + Bt$$

$$y'(t) = B$$

$$y'(0) = 0 \Rightarrow B = 0$$

$$y(t) = A + 0t = A = A \times 1$$

But then in this case $y'(1) = 0$ is automatically satisfied.

Conclusion: $\lambda = 0$ is an eigenvalue with eigenfunction $y(t) = 1$

Case 3: $\lambda < 0$

In this case $\lambda = -\omega^2$ where $\omega > 0$

Aux: $r^2 = \lambda = -\omega^2 \Rightarrow r = \pm\omega i$

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$y'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$y'(0) = -A\omega\sin(0) + B\omega\cos(0) = B\omega = 0 \Rightarrow B = 0$$

$$\begin{aligned} y(t) &= A\cos(\omega t) \\ y'(1) &= 0 \\ \cancel{-A\omega}\sin(\omega 1) &= 0 \\ \sin(\omega) &= 0 \\ \omega &= \pi m \qquad m = 1, 2, \cdots \end{aligned}$$

Here again m starts at 1 since we have $\omega > 0$

In that case $\lambda = -\omega^2 = -(\pi m)^2$ and $y = \cos(\omega t) = \cos(\pi m t)$

Note: If you let $m = 0$ then $\lambda = -(\pi 0)^2 = 0$ and $y = \cos(\pi 0 t) = \cos(0) = 1$, which *includes* the previous step!

Epic Conclusion:

Eigenvalues: $\lambda = -(\pi m)^2$ $(m = 0, 1, 2, \cdots) = 0, -\pi^2, -4\pi^2 \cdots$

Eigenfunctions:

$$y = \cos(\pi m t)\ (m = 0, 1, 2, \cdots) = 1, \cos(\pi t), \cos(2\pi t), \cdots$$

4. "Mixed" Example

Example 4: (extra practice)

Find the eigenvalues and eigenfunctions of

$$\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y'\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

Auxiliary Equation: $r^2 = \lambda$

Case 1: $\lambda > 0$

Then $r^2 = \lambda = \omega^2$ and so $r = \pm\omega$

$$y(t) = Ae^{\omega t} + Be^{-\omega t}$$
$$y(0) = A + B = 0 \Rightarrow B = -A$$
$$y(t) = Ae^{\omega t} - Ae^{-\omega t}$$
$$y'(t) = A\omega e^{\omega t} - A(-\omega)e^{-\omega t}$$

$$\begin{aligned} y'\left(\frac{\pi}{2}\right) &= 0 \\ A\omega e^{\frac{\pi\omega}{2}} + A\omega e^{-\frac{\pi\omega}{2}} &= 0 \\ \cancel{A\omega} e^{\frac{\pi\omega}{2}} &= -\cancel{A\omega} e^{\frac{\pi\omega}{2}} \\ \underbrace{e^{\frac{\pi\omega}{2}}}_{>0} &= \underbrace{-e^{\frac{\pi\omega}{2}}}_{<0} \end{aligned}$$

Which is a contradiction $\Rightarrow\Leftarrow$

Conclusion: In this case, we have no nonzero solutions

Case 2: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (repeated twice)

$$y(t) = A + Bt$$
$$y(0) = 0 \Rightarrow A = 0$$
$$y(t) = Bt$$

$$y'(t) = B$$

$$y'\left(\frac{\pi}{2}\right) = 0 \Rightarrow B = 0$$

But then in this case $y = 0 \Rightarrow\Leftarrow$

Conclusion: In this case, we have no nonzero solutions

Case 3: $\lambda < 0$

In this case $\lambda = -\omega^2$ where $\omega > 0$

Aux: $r^2 = \lambda = -\omega^2 \Rightarrow r = \pm\omega i$

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$y(0) = A = 0$$

$$y(t) = B\sin(\omega t)$$

$$y'(t) = \omega B\cos(\omega t)$$

$$\begin{aligned}
y'\left(\frac{\pi}{2}\right) &= 0 \\
\cancel{\omega B}\cos\left(\frac{\pi\omega}{2}\right) &= 0 \\
\cos\left(\frac{\pi\omega}{2}\right) &= 0 \\
\frac{\pi\omega}{2} &= \frac{\pi}{2} + \pi m \\
\omega &= 1 + 2m \qquad m = 0, 1, 2, \cdots
\end{aligned}$$

$$\lambda = -\omega^2 = -(2m+1)^2 = -1, -9, -25, \cdots$$

$$\text{And } y(t) = B\sin(\omega t) = B\sin\left((2m+1)t\right)$$

Epic Conclusion:

Eigenvalues:

$$\lambda = -\omega^2 = -\left(2m+1\right)^2 \ (m = 0, 1, 2, \cdots) = -1, -9, -25, \cdots$$

Eigenfunctions:

$$y = \sin\left(\left(2m+1\right)t\right) \ (m = 0, 1, 2\cdots) = \sin(t), \sin(3t), \sin(5t), \cdots$$

5. Inhomogeneous Equations

What if the right-hand-side of our ODE is nonzero?

Example 5: (Model Problem)

$$y'' - 5y' + 6y = 6e^{4t}$$

STEP 1: Homogeneous Solution

Solve the homogeneous equation $y'' - 5y' + 6y = 0$

$$y_0(t) = Ae^{2t} + Be^{3t}$$

STEP 2: Particular Solution

Find one solution of the original equation $y'' - 5y' + 6y = 6e^{4t}$

See next time, the result will be: $y_p(t) = 3e^{4t}$

STEP 3: Answer:

Fact:

The general solution is $y = y_0 + y_p$

$$y = Ae^{2t} + Be^{3t} + 3e^{4t}$$

Why this works?

On the one hand, can check that $y_0 + y_p$ is a sol. of $y'' - 5y' + 6y = 3e^{4t}$

On the other hand, given y and y_p, if you define $y_0 = y - y_p$ then can check that y_0 solves $y'' - 5y' + 6y = 0$, so y_0 is indeed a homogeneous solution, and $y_0 = y - y_p \Rightarrow y = y_0 + y_p$

Take-Away: To solve inhomogeneous equations, you first solve the homogeneous equation and then you find *one* solution of the original equation.

How to find y_p? We will discuss two methods. The first one is easier to apply but only works in special cases (see next time)