

MIDTERM I REVIEW

Content covers lectures 1-12 inclusive.

ODE CLASSIFICATION

- Linear or Nonlinear?
 - If the dependent variable (top of the differential) is not of the form, αy^n , where $n = 0, 1$, $\alpha = \text{constant}$, then the equation is nonlinear
- Homogeneity
 - If all terms contain the dependent variable, the equation is homogenous.
- Autonomous
 - Can be written in the form, $\frac{dy}{dt} = f(y)$
- Separable
 - Can be written in the form, $\frac{dy}{dt} = f(y)g(t)$

EXISTENCE AND UNIQUENESS

Consider the ODE

$$\begin{aligned}y' &= f(y, t) \\ y(t_0) &= y_0\end{aligned}$$

where y_0 is given.

- If f and $\frac{\partial f}{\partial y}$ are continuous, then the ODE has a unique solution $y = y(t)$ for t close enough to t_0 .

SOLVING SEPARABLE EQUATIONS

$$\frac{dy}{dx} = g(y)f(x)$$

1. Identify $g(y)$ and $f(x)$
2. Get like-terms to the same side and set up integrals
3. Solve the integrals
4. Combine the arbitrary constants, c
5. Using I.C, solve for C
6. Solve for $y(x)$ (if possible)

EXAMPLE

$x \frac{dy}{dx} - kx^2y = x^2$, Separate and find the general solution, assuming $k \neq 0$

- Rewrite as $\frac{dy}{dx} = (1 + ky)x$
- $\frac{dy}{(1+ky)} = xdx$
- $\frac{1}{k} \ln(1 + ky) = \frac{x^2}{2} + c$
- $\ln(1 + ky) = \frac{kx^2}{2} + kC$
- $1 + ky = e^{kC} * e^{\frac{kx^2}{2}}$
- $y = Ce^{\frac{kx^2}{2}} - \frac{1}{k}$

SOLVING LINEAR FIRST-ORDER EQUATIONS

$$\frac{dy}{dx} = -P(x)y + Q(x)$$

1. Write in standard form ($y' + P(x)y = Q(x)$)
2. Find the integrating factor, $\rho(x) = e^{\int P(x)dx}$
3. Find $\int \rho(x)Q(x)dx + c$
4. Divide step 4, by $\rho(x)$ to get: $y = \frac{\int \rho(x)Q(x)dx+c}{\rho(x)}$
5. Find c from I.C

EXAMPLE

$$xy' = -2y + 3x$$

1. $y' + \frac{2}{x}y = 3$

2. $\rho(x) = \exp(\int P(x)dx) = 2 \exp(\ln(x)) = x^2$

3. $\int \rho(x)Q(x) dx = \int 3x^2 dx = x^3 + C$

4. $y(x) = \frac{x^3+C}{P(x)} = x + \frac{C}{x^2}$

SOLVING EXACT EQUATIONS

$$Pdx + Qdy = 0$$

1. Write in the above standard form
2. Check if the equation is exact (conservative)
 - Verify $Q_x = P_y$
3. Find f such that $\nabla f = (P, Q)$
 - Find $f = \int P dx$ and $f = \int Q dy$
 - Pick constants of integration so that these are equal
4. The general solution is $f(x, y) = C$
5. Find C from initial conditions

EXAMPLE

- Determine if the following are exact. If so, solve the ODE.

1. $4xy + 1 + (2x^2 + \cos(y))y' = 0$

2. $3xy + y^2 + (x^2 + xy)y' = 0$

SOLUTION

1. $4xy + 1 + (2x^2 + \cos(y))y' = 0 = (4xy + 1)dx + (2x^2 + \cos(y))dy$

- $P = 4xy + 1, Q = 2x^2 + \cos(y)$
 - $P_y = Q_x \Rightarrow$ Exact ODE
 - $f = \int P dx = 2x^2y + x + g(y)$
 - $f = \int Q dy = 2x^2y + \sin(y) + h(x)$
 - $\Rightarrow f(x,y) = 2x^2y + x + \sin(y)$
 - General solution is $2x^2y + x + \sin(y) = c$

2. $3xy + y^2 + (x^2 + xy)y' = 0 = (3xy + y^2)dx + (x^2 + xy)dy$

- $P = 3xy + y^2, Q = x^2 + xy$
 - $P_y \neq Q_x \Rightarrow$ not exact

APPLICATION: NEWTONS LAW OF COOLING

This describes how an object will cool in environment with a constant ambient temperature.

$$\frac{dT}{dt} = -k(T - T_0)$$

- T_0 is the ambient temperature, T is the temperature.
- **General Solution***
 - $T(t) = T_0 + (T_{hot} - T_0)e^{-kt}$
 - *I would not recommend memorizing this as the type of questions being asked around this application don't require it.

APPLICATION: MIXTURE PROBLEM

Variables are:

- r_I and r_o , the rates of liquid in and out the chamber (units of vol/time).
 - If these rates are equal, we call this a steady-state flow
 - If $r_I > r_o$, this is called start-up, since the volume in the chamber is increasing
 - If $r_I < r_o$, this is called shutdown, since the volume in the chamber is decreasing.
- c_I is the concentration of the solute going in (units of mass/vol)
- c_{out} is the concentration of the solute going out.
 - We used the “mixed well” assumption to say that the concentration in the chamber is uniform, which means the concentration going out of the chamber will be equivalent to the concentration of the chamber overall.
 - As an equation; $c_{out} = \frac{\text{total mass of solute}}{\text{total volume}} = \frac{x}{V}$
- V is the volume of fluid in the chamber
- t is the independent variable of time
- $x(t)$ is the mass of solute (dependent var)



By using compartment modeling we can derive an equation to model how the mass of solute changes in time

$$\frac{dx}{dt} = \text{Solute in} - \text{Solute Out} = c_I r_I - c_{out} r_o = c_I r_I - \frac{x}{V} r_o$$

$$\text{Where, } V = V_o + (r_I - r_o)t$$

With this information, you can plug in the known values and solve as a normal linear first-order equation.

POPULATION MODELS AND LOGISTIC GROWTH

- The idea behind population models is to model how the size of a population changes over time.

$$\frac{dP}{dt} = \text{rate of births} - \text{rate of deaths}$$

- This idea lead to three different models for populations covered in lecture

- Exponential Model

- This assumes there's no intraspecies interaction.
- $\frac{dP}{dt} = kP \Rightarrow P = P_0 e^{kt}$
 - k is the rate of change of the population (births – deaths)
- The solution is just an exponential (hence the name)

EXPONENTIAL MODEL AND LOGISTIC MODEL

- Exponential Model with Harvesting

- This assumes there's no intraspecies interaction.

- $\frac{dP}{dt} = kP - r$

- k is the natural rate of change of the population
- r is a constant harvesting rate
- The solution is just an exponential but the sign of the exponent depends on P_0

- Logistic Model

- Take the interspecies interaction to be dependent on the population size (carry capacity)

- $\frac{dP}{dt} = kP \left(1 - \frac{P}{C}\right)$

- k is the same as above
- C is the carrying capacity
- The solution is a sigmoid (logistic).

LOGISTIC GROWTH EXAMPLE

- Note you will need to use partial fraction decomposition:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$
$$A(x+2) + B(x+1) = 1$$

Then group like terms and then solve the system

$$Ax + Bx = 0x \Rightarrow A + B = 0$$
$$2A + B = 1$$

Therefore, $A=1, B=-1$

Example 2: logistic / carrying capacity.

A population of rabbits is growing over time according to the equation:

$$\frac{dP}{dt} = kP(200-P)$$

At $t=0$, $P(0)=100$ and was growing @ $\frac{1}{P}$ rabbit per year. What is population in 60 years?

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A population of rabbits is growing over time according to the equation:

$$\frac{dP}{dt} = KP(200-P)$$

At $t=0$, $P(0)=100$ and was growing @ $\frac{1}{100}$ rabbit per year. What is population in 60 years?

Find K

$$\frac{dP}{dt} = 1 = KP(200-P) \quad P=100 \quad \frac{dP}{dt} = K(100)(200-100) = 1$$

$$K = \frac{1}{10000}$$

Find $P(t)$

$$\frac{dP}{dt} = KP(200-P) \quad \frac{dP}{P(200-P)} = K dt$$

partial fractions

$$\frac{A}{P} + \frac{B}{200-P} \quad A(200-P) + BP = 1 \quad A=B = \frac{1}{200}$$

$$\frac{1}{200} \int \left(\frac{1}{P} + \frac{1}{200-P} \right) dP = K dt \quad \frac{1}{200} [\ln(P) - \ln(200-P)] = tK + C$$

$$\ln\left(\frac{P}{200-P}\right) = \frac{200}{10000} t + C$$

$$\frac{P}{200-P} = e^{0.02t+C} = C e^{0.02t}$$

Finding C

$$\frac{100}{200-100} = C e^0 \quad C=1 \quad \frac{P}{200-P} = e^{0.02t}$$

$$P = 200e^{0.02t} - P e^{0.02t} \sim P + P e^{0.02t} = 200e^{0.02t}$$

$$P(1 + e^{0.02t}) = 200e^{0.02t}$$

$$P(t) = \frac{200e^{0.02t}}{1 + e^{0.02t}} \cdot \frac{e^{-0.02t}}{e^{-0.02t}}$$

$$p(t) = \frac{200}{e^{0.02t} + 1}$$

Finding $p(60)$

$$p(60) = \frac{200}{e^{1.2} + 1} = \frac{200}{1.30119} = 153.7 \text{ rabbits}$$

note: it moves toward carrying capacity.

CRITICAL POINTS

$$\frac{dx}{dt} = f(x)$$

- The equation above is the general form for a linear first-order ODE.
- Finding the critical points, x_c
 - Critical points are defined as the roots of $f(x)$.
- I. Finding the Stability
 - Partition the domain with critical points as the boundaries.
 - At each critical point, see how the sign of $f(x)$ changes. This sign change tells you the stability
 - + to – means it is stable
 - - to + means it is unstable
 - Same sign means it is semi stable. + to + means it is stable from below, - to - means it is stable from above
 - You can represent these signs in either a phase-line or bifurcation diagram
 - Alternative method: Derivative test (not recommended)
 - If $f'(x_c) < 0$, it is stable. > 0 , is unstable, $= 0$, use another method

LINEAR SECOND-ORDER ODES WITH CONSTANT COEFFICIENTS

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = F(t)$$

- Linear
 - The independent variable is not being raised to a power greater than 1, or being run through a function
- Second-order
 - The highest derivative order is 2
- Ordinary
 - There is only a 1 variable dependence
 - This is opposed to PDES, Partial differential equations, where there are partial derivatives since the dependent variable varies based off 2 or more variables
- Constant Coefficients
 - The coefficients of each of the LHS terms is constant

SOLVING THE HOMOGENEOUS CASE

$$ay'' + by' + cy = 0$$

1. Find the characteristic equation

- This will be in the form of; $ar^2 + br + c = 0$
- Replace each “y” with a “rⁿ,” where “n” is the order of the “y” term

2. Potential Cases (r_1, r_2 are roots of the characteristic equation)

1. r_1, r_2 are real and distinct

- Solution takes the form of $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$

2. r_1, r_2 are real and repeated

- Solution takes the form of $y(x) = C_1e^{r_1x} + C_2xe^{r_2x}$

3. r_1, r_2 are complex

- Solution takes the form of $y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$
 - Where α is the real part of your root, and β is the coefficient of the imaginary part of your root (where $\beta \geq 0$)

3. Plug in ICs and solve for arbitrary constants

EXAMPLE

$$y'' - 4y' - 5y = 0, y(0) = 5, y'(0) = 1$$

1. Obtain Characteristic Equation and Find Roots

- $r^2 - 4r - 5 \rightarrow r = 1, -5$

2. Identify the case

- 2 real and distinct roots implies the exponential case

• Write the general solution

- $y = C_1e^{-x} + C_2e^{5x}$

• Plug in I.Cs and solve for constants

- $y = C_1e^{-x} + C_2e^{5x}$

- $y' = -C_1e^{-x} + 5C_2e^{5x}$

- $5 = C_1 + C_2$ & $1 = -C_1 + 5C_2$

- Therefore $C_1 = 4, C_2 = 1$

- $y(x) = 4e^{-x} + e^{5x}$

EXAMPLE

$$y'' - 2y' + 2y = 0, y(0) = 1, y'(0) = 0$$

1. Obtain Characteristic Equation and Find Roots

- $r^2 - 2r + 2 \rightarrow r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

2. Identify the case

- Complex roots implies the sinusoidal case with exponential amplitude (Case 3)
- Write the general solution
 - $y(x) = e^x(C_1 \cos(x) + C_2 \sin(x))$
- Plug in I.Cs and solve for constants
 - $y = e^x(C_1 \cos(x) + C_2 \sin(x))$
 - $y' = e^x(-C_1 \sin(x) + C_2 \cos(x)) + e^x(C_1 \cos(x) + C_2 \sin(x))$
 - $1 = C_1$ & $0 = C_2 + C_1$
 - Therefore $C_1 = 1, C_2 = -1$
- $y(x) = e^x(\cos(x) - \sin(x))$

PRACTICE

1. $y'' + 2y' + y = 0, y(0) = 5, y'(0) = -3$

2. $y'' + 6y' + 10y = 0, y(0) = 4, y'(0) = -5$

3. Potential Cases (k_1, k_2 are roots of the characteristic equation)

1. $D > 0$

- Solution takes the form of $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

2. $D = 0$

- Solution takes the form of $y(x) = C_1 e^{r_1 x} + C_2 x e^{r_2 x}$

3. $D < 0$

- Solution takes the form of $y(x) = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x))$

- Where α is the real part of your root, and β is the coefficient of the imaginary part of your root (where $\beta \geq 0$)

SOLUTION

1) $y'' + 2y' + y = 0, y_0 = 5, y'_0 = -3$

$$r^2 + 2r + 1 = 0 \Rightarrow \frac{-2 \pm \sqrt{4-4}}{2} \rightarrow r = -1, \text{ multiplicity } 2$$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Plug in I.C.s

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} \Rightarrow 5 = C_1$$

$$y'(x) = -C_1 e^{-x} + C_2 e^{-x} - C_2 x e^{-x} \Rightarrow -3 = -C_1 + C_2$$

$$C_1 = 5, C_2 = 2$$

$$y(x) = 5e^{-x} + 2xe^{-x}$$

2) $y'' + 6y' + 10y = 0, y(0) = 4, y'(0) = -5$

$$r^2 + 6r + 10 = 0 \Rightarrow \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i$$

$$y(x) = e^{-3x} (C_1 \cos(x) + C_2 \sin(x))$$

Plug in I.C.s

$$y(x) = C_1 e^{-3x} \cos(x) + C_2 e^{-3x} \sin(x)$$

$$y'(x) = -3C_1 e^{-3x} \cos(x) - C_1 e^{-3x} \sin(x) + -3C_2 e^{-3x} \sin(x) + C_2 e^{-3x} \cos(x)$$

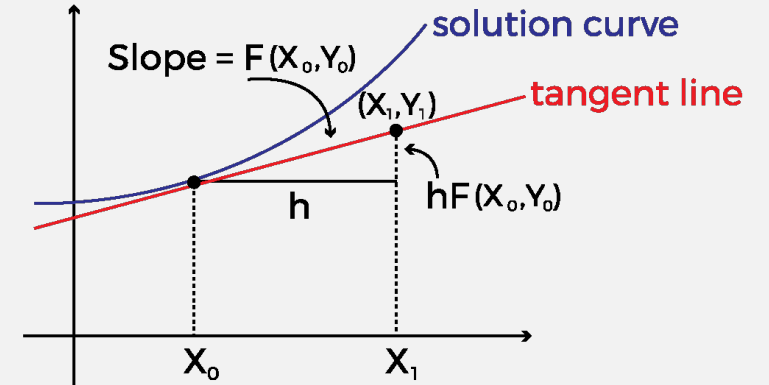
$$y(0) = 4 = C_1 + C_2 \Rightarrow C_1 = 4, C_2 = 7$$

$$y'(0) = -5 = -3C_1 + C_2$$

$$y(x) = e^{-3x} (4 \cos x + 7 \sin x)$$

NUMERICAL SCHEMES: EULER'S METHOD

- Forward Euler (Order 1)
 - Uses tangent line at x_0 to approximate



- Recursive update rule:

$$y_{n+1} = y_n + h f(y_n, t_n)$$

- $h = \frac{b-a}{N}$ is the step size, where N is the total number of steps
- Some notes
 - You need to pick a sufficiently small h for this to be a good approximation
 - The error order is $\mathcal{O}(Ch)$, which means the accuracy of the method scales inversely with the step size