# APMA 1941G Homework 1
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## Problem 1

Show that $p^1$ and $\boldsymbol{u}^1$ from the "acoustic approximation in fluid mechanics" in lecture satisfy the following PDE, where $c_0 \doteq \sqrt{g'(\rho^0)}$ and $\rho^0 \neq 0$

$$\begin{cases} p^1_{tt} - c_0^2 \Delta p^1 = 0 & \text{(i)} \\ \boldsymbol{u}^1_{tt} - c_0^2 \nabla(\text{div } \boldsymbol{u}^1) = 0 & \text{(ii)} \end{cases}.$$

**Proof.** First, recall from lecture that we found that

$$\begin{cases} \rho^0 \boldsymbol{u}^1_t = -\nabla p^1 & \text{(a)} \\ p^1 = g'(\rho^0)\rho^1 & \text{(b)} \\ \rho^1_t + \rho^0 \text{div}(\boldsymbol{u}^1) = 0 & \text{(c)} \end{cases}.$$

We will use these three identities derived in lecture to aid in showing that $p^1$ and $\boldsymbol{u}^1$ satisfy the system of PDEs. Differentiating (b) with respect to $t$, we find that

$$p^1_t = g'(\rho^0)\rho^1_t.$$

This is because $\rho^0$ is constant, so $g'(\rho^0)$ is a function evaluated at a constant and so is a constant (independent of $t$). Differentiating with respect to $t$ again, we then obtain

$$p^1_{tt} = g'(\rho^0)\rho^1_{tt} \tag{1}$$

Using the definition of $c_0$, (1) becomes

$$p^1_{tt} = c_0^2 \rho^1_{tt}. \tag{2}$$

Differentiating (c) with respect to $t$, we find that

$$\rho^1_{tt} + \rho^0 \text{div}(\boldsymbol{u}^1_t) = 0$$

Since $\rho^0$ is constant, we can pull it into the divergence operator, and we have

$$\rho^1_{tt} + \text{div}(\rho^0 \boldsymbol{u}^1_t) = 0. \tag{3}$$

Now substituting (a) into (3) and rearranging, we obtain

$$\rho^1_{tt} = \text{div}(\nabla p^1). \tag{4}$$

Substituting (4) into (2) and using the identity $\text{div}(\nabla p^1) = \Delta p^1$, we have shown the first PDE, (i), is satisfied, i.e.

$$p^1_{tt} - c_0^2 \Delta p^1 = 0.$$

To show the second PDE, (ii), is satisfied, we first take the gradient of (c) to find that

$$\nabla \rho^1_t + \rho^0 \nabla \text{div}(\boldsymbol{u}^1) = 0 \tag{5}$$

again using that $\rho^0$ is constant. Taking the time derivative of (b) gives that $g'(\rho^0)\rho^1_t = p^1_t$. Using the definition of $c_0$ and taking the gradient gives that $c_0^2 \nabla \rho^1_t = \nabla p^1_t$. Substituting this into (5), we have

$$\frac{1}{c_0^2} \nabla p^1_t + \rho^0 \nabla \text{div}(\boldsymbol{u}^1) = 0. \tag{6}$$

Multiplying through by $c_0^2$, we obtain

$$\nabla p^1_t + c_0^2 \rho^0 \nabla \text{div}(\boldsymbol{u}^1) = 0. \tag{7}$$

Taking the time derivative of (a), we have that $\nabla p^1_t = -\rho^0 \boldsymbol{u}^1_{tt}$. Substituting this identity into (7),

$$-\rho^0 \boldsymbol{u}^1_{tt} + c_0^2 \rho^0 \nabla \text{div}(\boldsymbol{u}^1) = 0. \tag{8}$$

Dividing through by $-\rho^0 \neq 0$, we have shown the second PDE, (ii), is satisfied, i.e.

$$\boldsymbol{u}^1_{tt} - c_0^2 \nabla(\text{div } \boldsymbol{u}^1) = 0.$$

## Problem 2

Suppose $u_0 = u_0(x)$ is a solution of

$$-u_0''(x) + V(x)u_0(x) = \lambda_0 u_0(x) \tag{9}$$

with $\lim_{|x|\to\infty} u_0(x) = 0$ where $x \in \mathbb{R}, \lambda_0 \in \mathbb{R}$ and $V(x)$ is given. Suppose we want to solve the following perturbation

$$-u_\epsilon''(x) + V(x)u_\epsilon(x) + \epsilon W(x)u_\epsilon(x) = \lambda_\epsilon u_\epsilon(x) \tag{10}$$

with $\lim_{|x|\to\infty} u_\epsilon(x) = 0$ where $W = W(x)$ is given.

### 2(a)

Expand $u_\epsilon$ and $\lambda_\epsilon$ out as

$$u_\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + ...$$
$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + ...$$

Plug this expansion into (10) and show that the $O(\epsilon^k)-$terms give you the following equation:

$$-u_k'' + (V - \lambda_0)u_k = -Wu_{k-1} + \sum_{j=1}^{k} \lambda_j u_{k-j}. \tag{11}$$

What do you get when you compare the O(1) terms?

Hint: the following formula might be useful:

$$(\sum_{k=0}^{\infty} a_k)(\sum_{k=0}^{\infty} b_k) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j}.$$

**Proof.** First, we plug in the expansion. We obtain

$$-u_0''-\epsilon u_1''-\epsilon^2 u_2''-...+V(u_0+\epsilon u_1+\epsilon^2 u_2+...)+\epsilon W(u_0+\epsilon u_1+...) = (\lambda_0+\epsilon\lambda_1+...)(u_0+\epsilon u_1+...) \tag{12}$$

Using the hint for the right hand side of (12), this becomes

$$(\sum_{k=0}^{\infty} \epsilon^k\lambda_k)(\sum_{k=0}^{\infty} \epsilon^k u_k) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \epsilon^j\lambda_j\epsilon^{k-j}u_{k-j} = \sum_{k=0}^{\infty}\epsilon^k\sum_{j=0}^{k}\lambda_j u_{k-j}.$$

Now considering the $O(\epsilon)$ terms, we have

$$-\epsilon u_1'' + \epsilon V u_1 = -\epsilon W u_0 + \epsilon\sum_{j=0}^{1}\lambda_j u_{1-j}$$

Dividing through by $\epsilon$, we have

$$-u_1'' + Vu_1 = -Wu_0 + \lambda_0 u_1 + \lambda_1 u_0,$$

or equivalently,

$$-u_1'' + (V - \lambda_0)u_1 = -Wu_0 + \lambda_1 u_0. \tag{13}$$

Directly analogously, we can see that comparing the $O(\epsilon^2)$ terms on each side yields

$$-\epsilon^2 u_2'' + V\epsilon^2 u_2 + \epsilon^2 W u_1 = \epsilon^2\sum_{j=0}^{2}\lambda_j u_{2-j},$$

which we can rearrange to obtain

$$-u_2'' + Vu_2 = -Wu_1 + \sum_{j=1}^{2}\lambda_j u_{2-j}. \tag{14}$$

Since all of the rest of the cases may be done in an exactly analogous fashion, to obtain directly analogous equations to (13) and (14), we can iteratively see that an equation of the desired form holds for each $k = 1, 2, ...$

**O(1) terms:** Comparing these terms on each side, we obtain

$$-u_0'' - Vu_0 + \epsilon W u_0 = \lambda_0 u_0,$$

which is exactly the original equation (9) that $u_0$ is a solution to.

**2(b)**

Let's look for solutions of the form

$$u_k(x) = u_0(x) w_k(x)$$

where $w_k$ is to be found. Plug this into (11) and multiply by $u_0$ and obtain

$$(u_0^2 w_k')' = u_0^2 \left( W w_{k-1} - \sum_{j=1}^{k} \lambda_j w_{k-j} \right). \tag{15}$$

**Solution.** First plugging in, noting that $-u_k' = -u_0' w_k - u_0 w_k'$ and $-u_k' = -u_0'' w_k - 2u_0' w_k' - u_0 w_k''$, we obtain

$$-u_0'' w_k - 2u_0' w_k' - u_0 w_k'' + (V - \lambda_0) u_0 w_k = -W u_0 w_{k-1} + \sum_{j=1}^{k} \lambda_j u_0 w_{k-j}.$$

Multiplying through by $u_0$ and collecting terms,

$$u_0(-u_0'' w_k - 2u_0' w_k' - u_0 w_k'' + (V - \lambda_0) u_0 w_k) = -u_0^2 \left( W w_{k-1} - \sum_{j=1}^{k} \lambda_j w_{k-j} \right) \tag{16}$$

Since $u_0$ solves $-u_0'' + Vu_0 = \lambda_0 u_0$, $u_0(V - \lambda_0) = u_0''$. Substituting this into (16), this term and the term $-u_0'' w_k$ sum to zero, so we then have

$$u_0(2u_0' w_k' - u_0 w_k'') = -u_0^2 \left( W w_{k-1} - \sum_{j=1}^{k} \lambda_j w_{k-j} \right). \tag{17}$$

Now we can notice that

$$(u_0^2 w_k')' = 2u_0 u_0' w_k' + u_0^2 w_k''$$

Using this identity in (17) and multiplying both sides by $-1$, we have shown that

$$(u_0^2 w_k')' = u_0^2 \left( W w_{k-1} - \sum_{j=1}^{k} \lambda_j w_{k-j} \right)$$

as required.

**2(c)**

Integrate (15) over $\mathbb{R}$, assuming that

$$u_0^2(x)w_k'(x) \to 0 \text{ as } |x| \to \infty \tag{18}$$

and solve for $\lambda_k$ to obtain the recursive definition of $\lambda_k$,

$$\lambda_k = \frac{\int_{-\infty}^{\infty} u_0(Wu_{k-1} - \sum_{j=1}^{k-1} \lambda_j u_{k-j})}{\int_{-\infty}^{\infty} u_0^2}.$$

**Solution.** Integrating both sides of (15) with respect to $x$, we have

$$\int_{-\infty}^{\infty} (u_0^2 w_k')' dx = \int_{-\infty}^{\infty} u_0^2 (Ww_{k-1} - \lambda_k w_0 - \sum_{j=1}^{k-1} \lambda_j w_{k-j}) dx. \tag{19}$$

First, we can see that the left-hand side is, for some arbitrary $a \in \mathbb{R}$ and using the Fundamental Theorem of Calculus,

$$\begin{aligned}
\lim_{x\to\infty} \int_a^x (u_0^2 w_k')' + \lim_{x\to-\infty} \int_x^a (u_0^2 w_k')' &= \lim_{x\to\infty} (u_0^2 w_k') - u_0(a)^2 w_k'(a) + u_0^2(a) w_k'(a) - \lim_{x\to-\infty} (u_0^2 w_k') \\
&= \lim_{x\to\infty} (u_0^2 w_k') - \lim_{x\to-\infty} (u_0^2 w_k') \\
&= 0 \text{ by the assumption (18).}
\end{aligned}$$

Thus (19) is now

$$0 = \int_{-\infty}^{\infty} u_0^2 (Ww_{k-1} - \lambda_k w_0 - \sum_{j=1}^{k-1} \lambda_j w_{k-j}) dx. \tag{20}$$

Since $w_{k-1} = u_{k-1}/u_0$, (20) becomes

$$0 = \int_{-\infty}^{\infty} u_0 (Wu_{k-1} - \sum_{j=1}^{k-1} \lambda_j u_{k-j}) - \int_{-\infty}^{\infty} \lambda_k u_0^2.$$

Since $\lambda_k$ is constant (we can make this assumption from the problem statements given that $\lambda_0 \in \mathbb{R}$ and none of the $\lambda_k$ appear with arguments), we can pull this out of the integral and rearrange to get

$$\lambda_k = \frac{\int_{-\infty}^{\infty} u_0(Wu_{k-1} - \sum_{j=1}^{k-1} \lambda_j u_{k-j})}{\int_{-\infty}^{\infty} u_0^2}$$

as required.

**2(d)**

Integrate (15) over $(-\infty, t)$ for $t > 0$, again assuming (18). Then integrate over $(-\infty, x)$ assuming $w_k(x)$ goes to zero at $-\infty$. Finally, use $u_k(x) = u_0(x)w_k(x)$ to conclude

$$u_k(x) = u_0(x) \int_{-\infty}^{1} \frac{1}{u_0^2(t)} \int_{-\infty}^{t} u_0(s) \left( W(s)u_{k-1}(s) - \sum_{j=1}^{k} \lambda_k u_{k-j}(s) \right) ds dt,$$

which is a recursive definition of $u_k(x)$ and an example of a Rayleigh-Schrodinger perturbation.

**Solution.** Integrating (15) over $(-\infty, t)$ for some $t > 0$, the left-hand side becomes (using FTC and our given assumption again)

$$\lim_{x\to-\infty} \int_x^t (u_0^2 w_k')' = u_0^2(t) w_k'(t) - \lim_{x\to-\infty} u_0^2(x) w_k'(x) = u_0^2(t) w_k'(t)$$

so we have overall

$$u_0^2(t)w_k'(t) = \int_{-\infty}^{t} u_0(s)^2 \left( W(s)w_{k-1}(s) - \sum_{j=1}^{k} \lambda_j w_{k-j}(s) \right) ds. \tag{21}$$

We now divide both sides by $u_0^2(t)$. Now integrating the resulting equation over $(-\infty, x)$, the resulting left-hand side is

$$\begin{aligned}\int_{-\infty}^{x} w_k'(t)dt &= w_k(x) - \lim_{y\to-\infty} w_k(y) \\ &= w_k(x) \text{ by the assumption on the limit of } w_k.\end{aligned}$$

Applying that $u_k(x) = u_0(x)w_k(x)$ so $w_k(x) = u_k(x)/u_0(x)$ to the resulting right hand side of (21), with the integration we obtain

$$\int_{-\infty}^{x} \frac{1}{u_0^2(t)} \int_{-\infty}^{t} u_0(s) \left( W(s)u_{k-1}(s) - \sum_{j=1}^{k} \lambda_k u_{k-j}(s) \right) dsdt.$$

Applying $w_k(x) = u_k(x)/u_0(x)$ to the left-hand side, which is now $w_k(x)$, and putting everything together, we have

$$\frac{u_k(x)}{u_0(x)} = \int_{-\infty}^{x} \frac{1}{u_0^2(t)} \int_{-\infty}^{t} u_0(s) \left( W(s)u_{k-1}(s) - \sum_{j=1}^{k} \lambda_k u_{k-j}(s) \right) dsdt.$$

Multiplying both sides by $u_0(x)$ yields

$$u_k(x) = u_0(x) \int_{-\infty}^{1} \frac{1}{u_0^2(t)} \int_{-\infty}^{t} u_0(s) \left( W(s)u_{k-1}(s) - \sum_{j=1}^{k} \lambda_k u_{k-j}(s) \right) dsdt,$$

as claimed.