APMA 1941G – HOMEWORK 2

Problem 1: (8 points)

Show that, in the “Derivation of the KdV equation” example in lecture, after the change of variable

$$\begin{cases} \xi := \sqrt{\epsilon}\,(x - c_0 t) \\ \tau := \epsilon^{\frac{3}{2}}\, t \\ \psi := \sqrt{\epsilon}\,\phi \end{cases}$$

Where $c^0 = \sqrt{h^0}$ the PDE for $\tilde{\psi}$ and $\tilde{h}$ become

$$\begin{cases} \epsilon \tilde{\psi}_{\xi\xi} + \tilde{\psi}_{yy} = 0 & \\ \tilde{\psi}_y = 0 & \text{at } y = 0 \\ \epsilon^2 \tilde{\psi}_\tau - \epsilon c_0 \tilde{\psi}_\xi + \frac{1}{2}\left(\epsilon(\tilde{\psi}_\xi)^2 + (\tilde{\psi}_y)^2\right) = -\epsilon(\tilde{h} - h^0) & \text{at } y = \tilde{h} \\ \epsilon^2 \tilde{h}_\tau + \epsilon(\tilde{\psi}_\xi - c_0)\tilde{h}_\xi = \tilde{\psi}_y & \text{at } y = \tilde{h} \end{cases}$$

Here $\tilde{\psi}$ and $\tilde{h}$ are ψ and h in the new variables, i.e.

$$\psi(x, y, t) = \tilde{\psi}(\xi, y, \tau) \text{ and } h(x, t) = \tilde{h}(\xi, \tau)$$

It may also be useful to use $\tilde{\phi}$ i.e. $\phi(x, y, t) = \tilde{\phi}(\xi, y, \tau)$

Note: For sake of clarity, the original PDE are

$$\begin{cases} \Delta\phi = 0 & \\ \phi_y = 0 & \text{at } y = 0 \\ \phi_t + \frac{1}{2}\left|\nabla\phi\right|^2 = -\left(h - h^0\right) & \text{at } y = h \\ h_t + \phi_x h_x = \phi_y & \text{at } y = h \end{cases}$$

Problem 2: (12 points, 4 points each)

Let u^ϵ and v^ϵ solve the system

$$\begin{cases} u^\epsilon_t + \frac{1}{\epsilon}u^\epsilon_x = \frac{(v^\epsilon)^2 - (u^\epsilon)^2}{\epsilon^2} \\ v^\epsilon_t - \frac{1}{\epsilon}v^\epsilon_x = \frac{(u^\epsilon)^2 - (v^\epsilon)^2}{\epsilon^2} \end{cases} \tag{1}$$

Let our Ansatz be:

$$u^\epsilon = u^0 + \epsilon u^1 + o(\epsilon)$$
$$v^\epsilon = v^0 + \epsilon v^1 + o(\epsilon)$$

And moreover suppose that $u^0 > 0$ and $v^0 > 0$

(a) Plug the Ansatz into (1) and show that

$$u^0 = v^0 \tag{2}$$
$$u^0_x = 2u^0(v^1 - u^1) \tag{3}$$

(b) Notice that if we add the two equations of (1), we get that

$$(u^\epsilon + v^\epsilon)_t + \frac{1}{\epsilon}(u^\epsilon - v^\epsilon)_x = 0 \tag{4}$$

Plug the Ansatz into (4) and show that

$$u_t^0 = \frac{1}{2}(v^1 - u^1)_x$$

(c) Using the results of (a) and (b) show that u^0 is a solution of the nonlinear heat equation

$$w_t - \frac{1}{4}(\ln w)_{xx} = 0$$

Hint: Use (3) to write $(v^1 - u^1)_x$ only in terms of u^0 and its derivatives. It may be helpful to write out explicitly what $(\ln w)_{xx}$ is