APMA 1941G Homework 2
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Problem 1

Show that, in the "Derivation of the KdV equation" example in lecture, after the change of variables,

$$\begin{cases} \xi \doteq \sqrt{\epsilon}(x - c_0 t) \\ \tau \doteq \epsilon^{3/2} t \\ \psi \doteq \sqrt{\epsilon}\phi \end{cases}$$

where $c^0 = \sqrt{h^0}$ the PDE for $\tilde{\psi}$ and $\tilde{h}$ become

$$\begin{cases} \epsilon\tilde{\psi}_{\xi\xi} + \tilde{\psi}_{yy} = 0 & \text{(i)} \\ \tilde{\psi}_y = 0 \quad \text{at } y = 0 & \text{(ii)} \\ \epsilon^2\tilde{\psi}_\tau - \epsilon c_0\tilde{\psi}_\xi + \frac{1}{2}\left(\epsilon(\tilde{\psi}_\xi)^2 + (\tilde{\psi}_y)^2\right) = -\epsilon(\tilde{h} - h^0) \quad \text{at } y = \tilde{h} & \text{(iii)} \\ \epsilon^2\tilde{h}_\tau + \epsilon(\tilde{\psi}_\xi - c_0)\tilde{h}_\xi = \tilde{\psi}_y \quad \text{at } y = \tilde{h} & \text{(iv)} \end{cases}$$

where $\psi(x, y, t) = \tilde{\psi}(\xi, y, \tau)$ and $h(x, t) = \tilde{h}(\xi, \tau)$, and $\tilde{\phi}$ similarly denotes ϕ in the new variables. The starting PDE is

$$\begin{cases} \Delta\phi = 0 & \text{(A)} \\ \phi_y = 0 \quad \text{at } y = 0 & \text{(B)} \\ \phi_t + \dfrac{1}{2}|\nabla\phi|^2 = -(h - h^0) \text{ at } y = h & \text{(C)} \\ h_t + \phi_x h_x = \phi_y \text{ at } y = h & \text{(D)} \end{cases}$$

Proof. First, write (A) in the new variables. We evaluate

$$\begin{aligned} \Delta\phi &= \phi_{xx} + \phi_{yy} \\ &= \tilde{\phi}_{\xi\xi} \cdot \sqrt{\epsilon} \cdot \sqrt{\epsilon} + \tilde{\phi}_{yy} \text{ chain rule} \\ &= \frac{1}{\sqrt{\epsilon}}(\epsilon\tilde{\psi}_{\xi\xi} + \tilde{\psi}_{yy}) \text{ definition of } \psi\text{, linearity of the derivative, and chain rule for } \xi \end{aligned}$$

Since $\Delta\phi = 0$, $\frac{1}{\sqrt{\epsilon}}(\epsilon\tilde{\psi}_{\xi\xi} + \tilde{\psi}_{yy}) = 0$. Thus

$$\epsilon\tilde{\psi}_{\xi\xi} + \tilde{\psi}_{yy} = 0,$$

yielding (i).

Next, we write (B) in the new variables. Since y is not transformed, we also have $\tilde{\phi}_y = 0$ at $y = 0$, which implies $\tilde{\psi}_y = 0$ at $y = 0$ since $\tilde{\psi} = \sqrt{\epsilon}\tilde{\phi}$. Thus, we also have (ii).

Now we write (C) in the new variables. First, by the chain rule (noting that the variable t is both in τ and ξ),

$$\phi_t = (\epsilon^{-1/2}\psi)_t = \epsilon^{-1/2}\psi_t = \epsilon^{-1/2}\epsilon^{3/2}\tilde{\psi}_\tau - \epsilon^{-1/2}c_0\epsilon^{1/2}\tilde{\psi}_\xi = \epsilon\tilde{\psi}_\tau - c_0\tilde{\psi}_\xi.$$

For the second term,

$$\frac{1}{2}|\nabla\phi|^2 = \frac{1}{2}(\epsilon^{-1/2})^2|\nabla\psi|^2 = \frac{1}{2}\epsilon^{-1}((\psi_x)^2 + (\psi_y)^2) = \frac{1}{2}\epsilon^{-1}\left((\sqrt{\epsilon})^2(\tilde{\psi}_\xi)^2 + (\tilde{\psi}_y)^2\right).$$

Adding together the two terms in the new variables and setting them equal to the right-hand side, we obtain

$$\epsilon\tilde{\psi}_\tau - c_0\tilde{\psi}_\xi + \frac{1}{2}\epsilon^{-1}\left(\epsilon(\tilde{\psi}_\xi)^2 + (\tilde{\psi}_y)^2\right) = -(\tilde{h} - h^0)$$

(valid at $y = \tilde{h}$, noting that the transformation in h did not necessitate any additional changes here because we are not differentiating and h^0 is a constant). Multiplying through by ϵ to clear the fraction, we obtain exactly (iii).

Lastly, we write (D) in the new variables. First,

$$h_t = \epsilon^{3/2}\tilde{h}_\tau - \epsilon^{1/2}c_0\tilde{h}_\xi.$$

Next,

$$\phi_x = \sqrt{\epsilon}\tilde{\phi}_\xi = \tilde{\psi}_\xi$$

and

$$h_x = \sqrt{\epsilon}\tilde{h}_\xi.$$

We then can replace the term

$$\phi_x h_x = \epsilon^{1/2}\tilde{\psi}_\xi\tilde{h}_\xi$$

Lastly,

$$\phi_y = \tilde{\phi}_y = \epsilon^{-1/2}\tilde{\psi}_y.$$

Putting this all together and multiplying through by $\epsilon^{1/2}$ in order to clear the fraction, we obtain

$$\epsilon^2\tilde{h}_\tau - \epsilon c_0\tilde{h}_\xi + \epsilon\tilde{\psi}_\xi\tilde{h}_\xi = \tilde{\psi}_y,$$

or equivalently,

$$\epsilon^2\tilde{h}_\tau + \epsilon(\tilde{\psi}_\xi - c_0)\tilde{h}_\xi = \tilde{\psi}_y$$

at $y = \tilde{h}$ which is (iv) as desired.

Problem 2

Let u^ε and v^ε solve the system

$$\begin{cases} u_t^\varepsilon + \frac{1}{\varepsilon} u_x^\varepsilon &= \frac{(v^\varepsilon)^2 - (u^\varepsilon)^2}{\varepsilon^2} \\ v_t^\varepsilon - \frac{1}{\varepsilon} v_x^\varepsilon &= \frac{(u^\varepsilon)^2 - (v^\varepsilon)^2}{\varepsilon^2} \end{cases} \tag{1}$$

Let our ansatz be:

$$u^\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon)$$
$$v^\varepsilon = v^0 + \varepsilon v^1 + o(\varepsilon)$$

and suppose that $u^0 > 0$ and $v^0 > 0$.

(a)

Plug the ansatz into (1) and show that

$$u^0 = v^0 \tag{2}$$
$$u_x^0 = 2u^0(v^1 - u^1) \tag{3}$$

Proof. First, plugging the ansatz into the first equation in the system (1), we obtain (omitting the $o(\cdot)$ terms as we will justify that this will not affect our comparisons)

$$u_t^0 + \epsilon u_t^1 + \frac{1}{\epsilon} u_x^0 + u_x^1 = \frac{(v^0)^2 - (u^0)^2}{\epsilon^2} + \frac{2(v^0 v^1 - u^0 u^1)}{\epsilon} + (v^1)^2 - (u^1)^2. \tag{4}$$

It is clear that the stand-alone $o(\cdot)$ terms will not directly affect the comparison as they will appear on both sides. However, on the right-hand side, we have omitted cross-terms involving u or v terms and the $o(\epsilon)$ terms,

$$(2v^0 o(\epsilon) + \epsilon 2 v^1 o(\epsilon)) - (2u^0 o(\epsilon) + \epsilon 2 u^1 o(\epsilon)).$$

Of these, the only ones that would be relevant to the following comparison is when we compare terms divided by ϵ, we could view $\frac{(2v^0 - 2u^0)o(\epsilon)}{\epsilon^2}$ as $\frac{2(v^0 - u^0)o(1)}{\epsilon}$, but that will be zero by what we find from a comparison of the terms on the order ϵ^{-2}, which none of these additional cross-terms are. We can compare the terms of order ϵ^{-2} first and this will not affect any of the other comparisons, and this will yield that the only relevant cross-terms are zero. The remaining cross-terms could be viewed as order 1 which will not be relevant with the following comparisons.

We may now proceed with the comparisons, and notice that there are no terms divided by ϵ^2 on the left-hand side of (4), so to match the order,

$$\frac{(v^0)^2 - (u^0)^2}{\epsilon^2} = 0$$

which implies

$$v^0 = u^0,$$

since $v^0, u^0 > 0$ by assumption. We have thus shown (2). To match the terms divided by ϵ, we have that

$$u_x^0 = 2(v^0 v^1 - u^0 u^1)$$

but since $u^0 = v^0$, this is equivalent to

$$u_x^0 = 2(u^0 v^1 - u^0 u^1) = 2u^0(v^1 - u^1),$$

which is (3).

(b)

Notice that if we add the two equations of (1), we get that

$$(u^\varepsilon + v^\varepsilon)_t + \frac{1}{\varepsilon}(u^\varepsilon - v^\varepsilon)_x = 0 \tag{5}$$

Plug the ansatz into (5) and show that

$$u_t^0 = \frac{1}{2}(v^1 - u^1)_x.$$

Proof. Again omitting the $o(\cdot)$ terms and plugging in, we obtain

$$u_t^0 + \epsilon u_t^1 + v_t^0 + \epsilon v_t^1 + \frac{1}{\epsilon}(u_x^0 + \epsilon u_x^1 - v_x^0 - \epsilon v_x^1) = 0.$$

Comparing the terms of order 1 after distributing the epsilon, we have

$$u_t^0 + v_t^0 + u_x^1 - v_x^1 = 0$$

Since we previously found that $u^0 = v^0$ though, it must be the case $u_t^0 = v_t^0$, so the above may be re-written

$$2u_t^0 = v_x^1 - u_x^1,$$

so we obtain

$$u_t^0 = \frac{1}{2}(v^1 - u^1)_x$$

as desired.

(c)

Using the results of (a) and (b), show that u^0 is a solution of the nonlinear heat equation

$$w_t - \frac{1}{4}(\ln w)_{xx} = 0. \tag{6}$$

Hint: Use (3) to write $(v^1 - u^1)_x$ only in terms of u^0 and its derivatives. It may be helpful to write out explicitly what $(\ln w)_{xx}$ is.

Proof. First, we follow the hint and rewrite

$$u_t^0 = \frac{1}{2}(v^1 - u^1)_x$$

but using the result of (a), $v^1 - u^1 = \frac{u_x^0}{2u^0}$, so

$$(v^1 - u^1)_x = \frac{2u^0 u_{xx}^0 - 2u_x^0 u_x^0}{4(u^0)^2} = \frac{1}{2}\left(\frac{u^0 u_{xx}^0 - (u_x^0)^2}{(u_0)^2}\right). \tag{7}$$

Thus, u^0 satisfies

$$u_t^0 - \frac{1}{4}\left(\frac{u^0 u_{xx}^0 - (u_x^0)^2}{(u_0)^2}\right) = 0. \tag{8}$$

Note that

$$(\ln w)_{xx} = \left(\frac{w_x}{w}\right)_x = \frac{w w_{xx} - w_x^2}{w^2},$$

where the first equality is due to the chain rule. Notice that (7) may accordingly be rewritten $\frac{1}{2}(\ln u^0)_{xx}$, using u^0 in place of w. Substituting this into (8), we obtain

$$u_t^0 - \frac{1}{4}(\ln u_0)_{xx} = 0,$$

showing that u^0 is indeed a solution of the nonlinear heat equation as claimed.