APMA 1941G – HOMEWORK 3

Problem 1: (10 points, 2 points each)

The purpose of this problem is to derive an explicit solution of a KdV equation. Consider the following equation

$$u_t + 6uu_x + u_{xxx} = 0$$

Where $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$

(a) Suppose u has the form

$u(x,t) = \phi(x - ct)$ for some $c \in \mathbb{R}$ and some $\phi = \phi(s)$

Plug u into the KdV equation to obtain that ϕ must satisfy

$$-c\phi' + 6\phi\,\phi' + \phi''' = 0 \text{ where } ' = \frac{d}{ds}$$

(b) Anti-differentiate the equation found in (a), multiply the resulting equation by ϕ' and anti-differentiate again to find that ϕ must satisfy

$$\frac{(\phi')^2}{2} = -\phi^3 + \frac{c}{2}\phi^2 + A\phi + B \text{ for some } A \text{ and } B$$

For the rest of the problem, we'll set $A = B = 0$, so we get

$$\frac{(\phi')^2}{2} = \phi^2\left(-\phi + \frac{c}{2}\right)$$

From which we get $\phi' = -\phi\sqrt{c - 2\varphi}$ (the $-$ is for convenience)

(c) Using separation of variables (like in your ODE course), show that we must have

$$s + C = -\int \frac{d\phi}{\phi\sqrt{c - 2\phi}} \text{ where } C \text{ is arbitrary}$$

(d) Use the substitution $\phi = \frac{c}{2}\operatorname{sech}^2(\theta)$ to show

$$s = \left(\frac{2}{\sqrt{c}}\right)\theta - C$$

Note: You can assume that $\frac{d\operatorname{sech}(\theta)}{d\theta} = -\operatorname{sech}(\theta)\tanh(\theta)$ and that $1 - \operatorname{sech}^2(\theta) = \tanh^2(\theta)$, and that here $\tanh(\theta) \geq 0$

(e) Use (d) and the definition of θ to conclude that

$$\phi(s) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(s + C)\right)$$

And finally conclude that

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct + C)\right)$$

Problem 2: (10 points, 2.5 points each)

The purpose of this problem is to prove Stirling's formula, which is very useful in probability; for example, it is used in the proof of the Central Limit Theorem:

$$n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \text{ as } n \to \infty,$$

Here $f \sim g$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, or equivalently $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 1$

(a) Show using induction on $n = 1, 2, \cdots$ and integration by parts that (you can assume that the terms at $t = \infty$ are 0)

$$\int_0^\infty e^{-t} t^{n-1} dt = (n-1)!$$

Note: If you're unfamiliar with induction, please check out:

Video: Induction Basic Example

Video: Induction With Functions

(b) Use the substitution $s = \frac{t}{n}$ to show the above integral equals

$$n^n \int_0^\infty e^{-n(s-\ln(s))} \frac{1}{s} ds$$

(c) The general Laplace method says that:

If ϕ is a smooth function that has a max at x_0 with $\phi'(x_0) = 0$ and $\phi''(x_0) < 0$, and $a(x)$ is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$

$$\int_0^\infty a(x) e^{\frac{\phi(x)}{\epsilon}} dx = \sqrt{\frac{2\pi\epsilon}{|\phi''(x_0)|}}\, e^{\frac{\phi(x_0)}{\epsilon}}\, a(x_0) \left(1 + o(1)\right)$$

Apply the general Laplace method to the integral in (b) with $\epsilon = \frac{1}{n}$ as well as the result in (a) to show that as $n \to \infty$

$$\frac{(n-1)!}{n^n} = \sqrt{2\pi} n^{-\frac{1}{2}} e^{-n} \left(1 + o(1)\right)$$

(d) Use the identity in (c) and the definition of $\sim$ given at the beginning of the problem to conclude that

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}$$