APMA 1941G Homework 3

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Problem 1

The purpose of this problem is to derive an explicit solution of a KdV equation. Consider the following equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

where $u = u(x, t)$, $x \in \mathbb{R}$, $t > 0$.

(a)

Suppose u has the form $u(x, t) = \phi(x - ct)$ for some $c \in \mathbb{R}$ and some $\phi = \phi(s)$. Plug u into the KdV equation to obtain that ϕ must satisfy

$$-c\phi' + 6\phi\phi' + \phi''' = 0. \tag{2}$$

Derivation. First, using the chain rule,

$$u_t = \frac{\partial}{\partial t}\phi(x - ct) = -c\phi'$$

where the $'$, as in the problem statement, denotes the derivative with respect to the parameter of ϕ, which is s. Similarly, $u_x = \phi'$ and $u_{xxx} = \phi'''$. Making these substitutions, along with $u = \phi$, in (**??**), we obtain

$$-c\phi' + 6\phi\phi' + \phi''' = 0.$$

Thus, since u satisfies (1), we can find that by making the ansatz that $u(x, t)$ has the form $\phi(x - ct)$, it satisfies exactly (2).

(b)

Anti-differentiate the equation found in (a), multiply the resulting equation by ϕ' and anti-differentiate again to find that ϕ must satisfy

$$\frac{(\phi')^2}{2} = -\phi^3 + \frac{c}{2}\phi^2 + A\phi + B \tag{3}$$

for some A and B.

Derivation. First, note that

$$\phi\phi' = (\frac{1}{2}\phi^2)'.$$

Now anti-differentiating (2) with respect to s, we obtain

$$-c\phi + 3\phi^2 + \phi'' + A = 0.$$

Here, the coefficient of 3 came from $6 \cdot \frac{1}{2}$, and A is an arbitrary constant. Now multiplying through by ϕ', noticing that $3\phi^2\phi' = (\phi^3)'$ and $\phi''\phi' = (\frac{1}{2}(\phi')^2)'$ and anti-differentiating again,

$$\phi^3 - \frac{c}{2}\phi^2 + \frac{(\phi')^2}{2} + A\phi + B = 0.$$

Re-arranging, and re-labelling $-A$ and $-B$ as A and B (which we may do as they are arbitrary constants), we obtain exactly (3).

(c)

We set A and B to be zero in the above, so we obtain

$$\phi' = -\phi\sqrt{c - 2\phi}. \tag{4}$$

Using separation of variables, show that

$$s + C = -\int \frac{d\phi}{\phi\sqrt{c - 2\phi}} \tag{5}$$

where C is arbitrary.

Derivation. Beginning with (4) and writing it in a more suggestive form,

$$\frac{d\phi}{ds} = -\phi\sqrt{c - 2\phi}.$$

Re-arranging this and integrating both sides (as in separation of variables), we obtain

$$-\int \frac{d\phi}{\phi\sqrt{c - 2\phi}} = \int ds.$$

The right-hand side evaluates to $s + C$ for an arbitrary constant C, yielding exactly (5) as required.

(d)

Use the substitution $\phi = \frac{c}{2}\text{sech}^2(\theta)$ to show

$$s = \left(\frac{2}{\sqrt{c}}\right)\theta - C \tag{6}$$

Note that you can assume that $\frac{d\text{sech}(\theta)}{d\theta} = -\text{sech}(\theta)\text{tanh}(\theta)$ and that $1 - \text{sech}^2(\theta) = \text{tanh}^2(\theta)$, and that here $\text{tanh}(\theta) \geq 0$.

Derivation. Using the given substitution in (5), we are essentially doing u-substitution, so we account for the fact that $d\phi = \frac{c}{2} \cdot 2\text{sech}(\theta) \cdot -\text{sech}(\theta)\text{tanh}(\theta)d\theta = -c\text{sech}^2(\theta)\text{tanh}(\theta)$. We then obtain

$$s + C = \int \frac{c\text{sech}^2(\theta)\text{tanh}(\theta)}{\frac{c}{2}\text{sech}^2(\theta)\sqrt{c(1 - \text{sech}^2(\theta))}} d\theta.$$

Simplifying and using the identity $1 - \text{sech}^2(\theta) = \text{tanh}^2(\theta)$, we obtain

$$s + C = \int \frac{2\text{tanh}(\theta)}{\sqrt{c}\text{tanh}(\theta)} d\theta.$$

Note that here we used the assumption that $\text{tanh}(\theta) \geq 0$. We can simplify further to

$$s + C = \int \frac{2}{\sqrt{c}} d\theta.$$

Now completing the integration on the right hand side (and noting that we obtain another arbitrary constant of integration, so the following C is technically not the same C though this does not matter as it is arbitrary)

$$s + C = \left(\frac{2}{\sqrt{c}}\right)\theta.$$

Re-arranging this immediately yields (6) as required.

(e)

Use (d) and the definition of θ to conclude that

$$\phi(s) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(s+C)\right) \tag{7}$$

and finally conclude that

$$u(x,t) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct+C)\right). \tag{8}$$

Conclusion. First re-arranging (6) to obtain θ in terms of s, we have

$$\theta = \frac{\sqrt{c}}{2}(s+C).$$

Since $\phi(s) = \frac{c}{2}\text{sech}^2(\theta)$, then

$$\phi(s) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(s+C)\right),$$

which verifies (7). Since at the outset we made the ansatz $u(x,t) = \phi(x-ct)$, we replace s with $x-ct$, and obtain

$$u(x-ct) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct+C)\right),$$

verifying (8).

Problem 2

The purpose of this problem is to prove Stirling's formula, or that $n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$ as $n \to \infty$.

(a)

Show using induction on $n = 1, 2, ...$ and integration by parts that (assuming the terms at $t = \infty$ are 0)

$$\int_0^\infty e^{-t} t^{n-1} dt = (n-1)!. \tag{9}$$

Proof. *Base Case.* We first show the required statement is true for $n = 1$. Plugging $n = 1$ into (9), we obtain $\int_0^\infty e^{-t} dt$ on the left hand side and $0! = 1$ on the right hand side. Indeed, these are equal, as the left hand side evaluates to $\lim_{x\to\infty}(-e^{-x}) + e^0 = 1$.
Inductive Hypothesis. Suppose that (9) is true for some arbitrary $n \in \mathbb{N}$.
Inductive Step. Given the inductive hypothesis is true, we aim to now show that (9) holds for $n+1$, where this is the (arbitrary) n from the inductive hypothesis. Considering (9) for $n+1$ rather than n, what we aim to show is that

$$\int_0^\infty e^{-t} t^n dt = n!.$$

Using integration by parts for the integral, where we are using $u = t^n$ and $dv = e^{-t} dt$,

$$\begin{aligned}
\int_0^\infty e^{-t} t^n dt &= -e^{-t} t^n \, |_0^\infty + n \int_0^\infty e^{-t} t^{n-1} dt \\
&= n \int_0^\infty e^{-t} t^{n-1} dt \text{ due to our assumption and } 0^n = 0 \text{ for } n = 1, 2, ... \\
&= n(n-1)! \text{ inductive hypothesis} \\
&= n!
\end{aligned}$$

Thus, we have shown the inductive step. Therefore, we may conclude that the desired identity holds for all $n = 1, 2, ...$

(b)

Use the substitution $s = \frac{t}{n}$ to show the integral in (9) equals

$$n^n \int_0^\infty e^{-n(s-\ln(s))} \frac{1}{s} ds.$$

Proof. Since $s = t/n$, $ds = \frac{1}{n} dt$, and we may rewrite (noting the bounds are still the same, as we are just scaling by a constant factor)

$$\begin{aligned}
\int_0^\infty e^{-t} t^{n-1} dt &= \int_0^\infty e^{-ns} (sn)^{n-1} \frac{1}{n} ds \\
&= n^n \int_0^\infty e^{-ns} s^{n-1} ds \\
&= n^n \int_0^\infty e^{-n(s-\ln(s))} \frac{1}{s} ds
\end{aligned}$$

as claimed.

(c)

The general Laplace method says that: If ϕ is a smooth function that has a max at x_0 with $\phi'(x_0) = 0$ and $\phi''(x_0) < 0$, and $a(x)$ is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$

$$\int_0^\infty a(x) e^{\frac{\phi(x)}{\epsilon}} dx = \sqrt{\frac{2\pi\epsilon}{|\phi''(x_0)|}} e^{\frac{\phi(x_0)}{\epsilon}} a(x_0)(1 + o(1)).$$

Apply the general Laplace method to the integral in (b) with $\epsilon = \frac{1}{n}$ as well as the result in (a) to show that as $n \to \infty$,

$$\frac{(n-1)!}{n^n} = \sqrt{2\pi} n^{-\frac{1}{2}} e^{-n}(1 + o(1)).$$

Proof. First, we justify our application of Laplace's method. We identify $\epsilon \doteq \frac{1}{n}$, so that $\epsilon \to 0$ is equivalent to $n \to \infty$. Next, we identify $a(x)$ with $\frac{1}{s}$ (the variable of integration is s), and $\phi(x)$ with $-(s - \ln(s))$. Indeed, $\frac{1}{s}$ is smooth on $(0, \infty)$ (the singularity at zero should not be an issue because the integral is still well-defined and we will not be evaluating a at zero, as we will see). Similarly, $-(s - \ln(s))$ is smooth on $(0, \infty)$. To maximize this function, we differentiate and set $-1 + \frac{1}{s} = 0$, so we find $s = 1$ is the maximizer. The other properties in the hypothesis of the function follow immediately, and we have also verified $x_0 \neq 0$.

Now, applying the method, we have that

$$\int_0^\infty e^{-n(s-\ln(s))} \frac{1}{s} ds = \sqrt{\frac{2\pi}{|\phi''(1)|n}} e^{n\phi(1)} a(1)(1 + o(1))$$

We compute: $\phi(1) = -1$, $\phi''(1) = -(1)^{-2} = -1$, $a(1) = 1$, hence

$$\int_0^\infty e^{-n(s-\ln(s))} \frac{1}{s} ds = \sqrt{\frac{2\pi}{n}} e^{-n}(1 + o(1))$$

Using the result from (a) to replace the integral on the left, noting that due to (b) it is in particular equal to $\frac{(n-1)!}{n^n}$, we obtain

$$\frac{(n-1)!}{n^n} = \sqrt{2\pi} n^{-1/2} e^{-n}(1 + o(1))$$

as required.

(d)

Multiply the above identity by n on both sides and conclude using the definition of $\sim$ that

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}.$$

Conclusion. Multiplying through by n, since $n(n-1)! = n!$,

$$\frac{n!}{n^n} = \sqrt{2\pi} n^{\frac{1}{2}} e^{-n}(1 + o(1)).$$

Multiplying through by n^n,

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}(1 + o(1)).$$

We can see that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}} = \lim_{n\to\infty} 1 + o(1) = 1$$

since a function $f(n)$ being $o(1)$ means $\lim_{n\to\infty} f(n) = 0$. Thus, according to the definition of $\sim$, we may conclude

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}.$$