APMA 1941G – HOMEWORK 6

Problem 1: (10 = 2 + 4 + 4 points)

Consider the equation

$$\begin{cases} u_\epsilon'' + \epsilon u_\epsilon' + u_\epsilon = 0 \\ u_\epsilon(0) = 1, u_\epsilon'(0) = 1 \end{cases} \tag{1}$$

Apply the Ansatz

$$u_\epsilon(t) = u_0(t, \epsilon t) + \epsilon u_1(t, \epsilon t) + \cdots$$

Here $u_\epsilon = u_\epsilon(t)$ and $u_k = u_k(t, \tau)$ (here τ is a placeholder for ϵt)

(a) Show that $u^0(t, \tau) = A(\tau)\cos(t) + B(\tau)\sin(t)$

(b) Find an ODE for A and B and solve it, and use that to solve for $u_0(t) = u_0(t, \epsilon t)$. Impose the conditions $u_0(0) = 1$ and $u_0'(0) = 1$

Hint: Select A and B to kill the resonance terms $\cos(t)$ and $\sin(t)$, just like we did for Duffing's equation

(c) Find the exact solution of the original ODE (1) and compare it with the solution u_0 you found in (b). Do we have

$$\lim_{\epsilon \to 0} u^\epsilon(t) = \lim_{\epsilon \to 0} u_0(t)?$$

Problem 2: (10 = 5 + 5 points) [The Inverted Pendulum Problem]

Consider the equation

$$\theta_\epsilon'' - \left[a + \frac{b}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)\right]\sin(\theta_\epsilon) = 0$$

Where $a > 0$ and $b > 0$ are constants, and $\theta = \theta(t)$

Apply the Ansatz

$$\theta_\epsilon(t) = \theta^0\left(t, \frac{t}{\epsilon}\right) + \epsilon\,\theta^1\left(t, \frac{t}{\epsilon}\right) + \cdots$$

Where $\theta^k = \theta^k(t,\tau)$, and $\tau \mapsto \theta^k(t,\tau)$ is 2π periodic

(a) Show that θ does not depend on τ, that is $\theta^0 = \theta^0(t)$

Hint: For the sin −term, use a (first-order) Taylor expansion at $\theta = \theta^0$ and $h = \epsilon\theta^1$. For the $\cos\left(\frac{t}{\epsilon}\right)$-term, leave it as it is and write it as $\cos(\tau)$ (in particular, do **NOT** expand it out!) Now once you get the $O\left(\frac{1}{\epsilon^2}\right)$ −term, multiply your result by θ^0 and integrate by parts with respect to τ on $[0, 2\pi]$, similarly to what we did in Example 1 (Homogenization). Why are there no boundary terms in the integration?

(b) Show that θ^0 satisfies the ODE

$$\theta^0_{tt} + \frac{b^2}{4}\sin(2\theta^0) - a\sin(\theta^0) = 0$$

Hint: Here find the $O\left(\frac{1}{\epsilon}\right)$ −terms and solve for θ^1. Your solution may involve constants $A = A(t)$ and $B = B(t)$. Set them equal to 0. Then, find the $O(1)-$terms and integrate whatever you found with respect to τ from 0 to 2π. Also remember your formula for θ^1, and use that $\int_0^{2\pi} \cos^2(x) dx = \pi$ and that $\sin(2x) = 2\sin(x)\cos(x)$