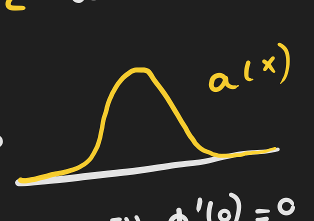
LAPLACE METHOD

GOAL STUDY $I[\varepsilon] = \int_{-\infty}^{\infty} a(x)\, e^{\frac{\phi(x)}{\varepsilon}}\, dx$

WHERE 1) $a(x)$ SMOOTH + COMPACT SUPP



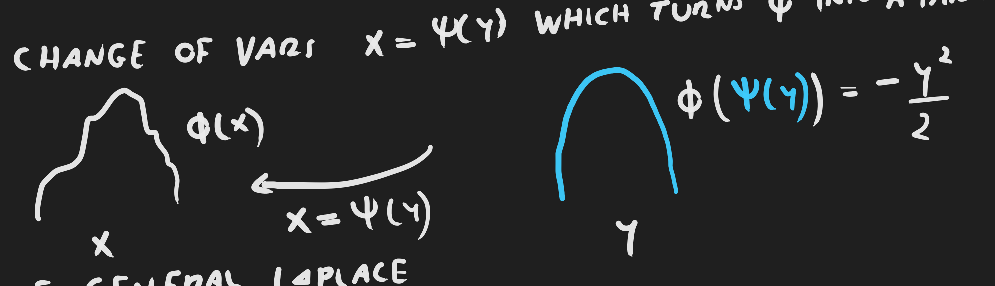
2) ϕ HAS A UNIQUE GLOBAL MAX AT 0 WITH $\phi'(0) = 0$, $\phi''(0) < 0$

3) $\phi(0) = 0$

THEOREM $I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(0)\, \varepsilon^{\frac{k+1}{2}}$

WHERE $(L_{2k}a)(0) = b_0 a(0) + b_1 a'(0) + \cdots + b_{2k} a^{(2k)}(0)$

IDEA CHANGE OF VARS $x = \psi(y)$ WHICH TURNS ϕ INTO A PARABOLA

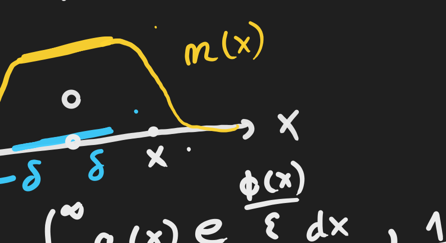


$\phi(\psi(y)) = -\frac{y^2}{2}$

PROOF OF GENERAL LAPLACE

STEP 1 LET ψ (AND U AND V) BE GIVEN BY THE MORSE LEMMA (LAST TIME)

LET m BE A SMOOTH SUPPORT FUNCTION WITH $m \equiv 0$ OUTSIDE V AND $m \equiv 1$ ON $(-\delta, \delta) \subseteq V$ FOR SOME $\delta > 0$



DECOMPOSE: $I[\varepsilon] = \int_{-\infty}^{\infty} a(x)\, e^{\frac{\phi(x)}{\varepsilon}}\, dx$, $1 = m + (1-m)$

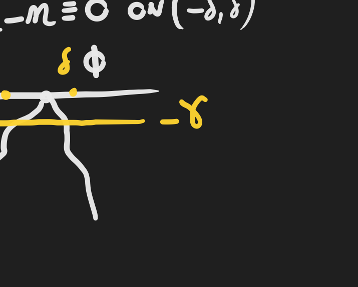
$$= \underbrace{\int_{-\infty}^{\infty} m\, a(x)\, e^{\frac{\phi}{\varepsilon}}\, dx}_{A} + \underbrace{\int_{-\infty}^{\infty} (1-m)\, a(x)\, e^{\frac{\phi}{\varepsilon}}\, dx}_{B}$$

WHAT WE WANT → A

B ← EXP SMALL

STEP 2 **STUDY OF B** $m \equiv 1$ ON $(-\delta, \delta) \Rightarrow 1 - m \equiv 0$ ON $(-\delta, \delta)$

SO WE ONLY CARE ABOUT $x \notin (-\delta, \delta)$



AND $\phi(x) \leq -\gamma$ ON $\mathbb{R} \setminus (-\delta, \delta)$ FOR SOME $\gamma > 0$

AND SO $e^{\phi/\varepsilon} \leq e^{-\gamma/\varepsilon}$ ← EXP SMALL

HENCE, SIMILARLY TO LAST TIME, $|B|$ EXP SMALL

STEP 3 STUDY OF A

$$A = \int_{\mathbb{R}} m(x)\, a(x)\, e^{\frac{\phi(x)}{\varepsilon}}\, dx$$

$x = \psi(y) \Rightarrow y = \psi^{-1}(x)$, $dx = \psi'(y)\, dy$

$$= \int_{\psi^{-1}(\mathbb{R})} m(\psi(y))\, a(\psi(y))\, e^{\frac{\phi(\psi(y))}{\varepsilon}}\, \psi'(y)\, dy$$

m HAS SUPP IN V

$$= \int_{\mathbb{R}} \underbrace{a(\psi(y))\, m(\psi(y))\, \psi'(y)}_{=:\, \tilde{a}(y)}\, e^{-\frac{y^2}{2\varepsilon}}\, dy$$

$$= \int_{-\infty}^{\infty} \tilde{a}(y)\, e^{-\frac{y^2}{2\varepsilon}}\, dy$$ ← BACK TO SPECIAL CASE!

$$\sim \sum_{k=0}^{\infty} \underbrace{\frac{(\tilde{a})^{(k)}(0)}{k!}\, c_k}_{=:\, (L_{2k}a)(0)}\, \varepsilon^{\frac{k+1}{2}}$$

$$\Rightarrow I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(0)\, \varepsilon^{\frac{k+1}{2}} \quad \blacksquare$$

EX LET'S CALCULATE THE FIRST TERM

$$I[\varepsilon] \sim L_0(a)(0)\sqrt{\varepsilon} + \underbrace{L_2(a)(0)\,\varepsilon^{\frac{3}{2}} + L_4(a)(0)\,\varepsilon^{\frac{5}{2}}}_{o(\sqrt{\varepsilon})}$$

$$= L_0(a)(0)\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$$

NOW $L_0(a)(0) \overset{DEF}{=} \frac{\tilde{a}(0)}{0!}\, c_0 \overset{DEF}{=} a(\psi(0))\, m(\psi(0))\, \psi'(0)\, c_0$ (*)

$$c_0 \overset{DEF}{=} \int_{-\infty}^{\infty} y^0\, e^{-\frac{y^2}{2}}\, dy = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy = \sqrt{2\pi} \text{ (GAUSSIAN INTEGRAL)}$$

⟹ WHAT ARE $\psi(0)$ AND $\psi'(0)$?

$\psi(0)$ START WITH $\phi(\psi(y)) = -y^2/2$

$\overset{y=0}{\Longrightarrow} \phi(\psi(0)) = 0$

$\Longrightarrow \boxed{\psi(0) = 0}$ B/C $\phi(y) = 0 \iff y = 0$

$\psi'(0)$ $\phi(\psi(y)) = -y^2/2$ ↓ DIFF

$\phi'(\psi(y))\,\psi'(y) = -y$ (**) ↓ $y = 0$

$\phi'(\underbrace{\psi(0)}_{0})\,\psi'(0) = -0$

$\underbrace{\phi'(0)}_{0}\,\psi'(0) = 0 \Rightarrow 0 = 0$???

TRICK DIFF $\phi'(\psi(y))\,\psi'(y) = -y$ AGAIN!

$\phi''(\psi(y))\,\psi'(y)\,\psi'(y) + \phi'(\psi(y))\,\psi''(y) = -1$ ↓ $y = 0$

$\phi''(\psi(0))\,(\psi'(0))^2 + \underbrace{\phi'(\underbrace{\psi(0)}_{0})}_{\phi'(0) = 0}\,\psi''(0) = -1$

$\Rightarrow \phi''(0)\,(\psi'(0))^2 = -1$

$\Rightarrow (\psi'(0))^2 = -1/\phi''(0)$

$\Rightarrow \psi'(0) = \sqrt{\frac{-1}{\phi''(0)}} = \frac{1}{\sqrt{|\phi''(0)|}}$

BACK TO $L_0(a)(0) = a(\psi(0))\, m(\psi(0))\, \psi'(0)\, c_0$

$$= a(0)\, \underbrace{m(0)}_{1}\, \frac{1}{\sqrt{|\phi''(0)|}}\, \sqrt{2\pi}$$

$$= a(0)\sqrt{\frac{2\pi}{|\phi''(0)|}}$$

$$\Longrightarrow I[\varepsilon] \sim a(0)\sqrt{\frac{2\pi\varepsilon}{|\phi''(0)|}} + o(\sqrt{\varepsilon}) \quad \leftarrow \text{EXPLICIT!}$$

1) WHAT IF $\phi(0) \neq 0$

TRICK $I[\varepsilon] = \int_{-\infty}^{\infty} a(x)\, e^{\frac{\phi(x)}{\varepsilon}}\, dx = \int_{-\infty}^{\infty} a\, e^{\frac{\phi(x) - \phi(0) + \phi(0)}{\varepsilon}}$

$$= e^{\frac{\phi(0)}{\varepsilon}} \int_{-\infty}^{\infty} a(x)\, e^{\frac{\phi(x) - \phi(0)}{\varepsilon}}\, dx$$

→ SATISFIES $\phi(0) = 0$

2) IF ϕ ATTAINS A MAX AT x_0 INSTEAD OF 0:

$$I[\varepsilon] = \int_{-\infty}^{\infty} a(x)\, e^{\frac{\phi(x)}{\varepsilon}}\, dx = \int_{-\infty}^{\infty} a(y + x_0)\, e^{\frac{\phi(y + x_0)}{\varepsilon}}\, dy$$

↑ $y = x - x_0$

ATTAINS A MAX AT $y = 0$

⟹ **FACT** IF ϕ ATTAINS A GLOBAL MAX AT x_0 (AND ONLY AT x_0) WITH $\phi'(x_0) = 0$ AND $\phi''(x_0) < 0$ THEN

$$I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(x_0)\, e^{\phi(x_0)/\varepsilon}\, \varepsilon^{\frac{k+1}{2}} = \sqrt{\frac{2\pi\varepsilon}{|\phi''(x_0)|}}\, a(x_0)\, e^{\frac{\phi(x_0)}{\varepsilon}} + o(\sqrt{\varepsilon})$$

APP STIRLING'S FORMULA

C - MULTIDIMENSIONAL CASE

MULTI-INDEX NOTATION:

$\underline{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$

$\underline{\alpha} = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$ ← MULTI-INDEX

$|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_N$ ← ORDER OF INDEX

$\underline{\alpha}! = \alpha_1! \cdots \alpha_N!$

$\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$

$D^{\underline{\alpha}} = \partial^{|\underline{\alpha}|} / \partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}$

EX $(N = 3)$ $\underline{\alpha} = (1, 2, 4)$

$|\underline{\alpha}| = 1 + 2 + 4 = 7$ ← ORDER

$\underline{\alpha}! = 1!\, 2!\, 4! = 48$

$\underline{x}^{\underline{\alpha}} = x_1^1 x_2^2 x_3^4$

$D^{\underline{\alpha}} f = D^{(1,2,4)} f = f_{\underbrace{x_1}_{1} \underbrace{x_2 x_2}_{2} \underbrace{x_3 x_3 x_3 x_3}_{4}}$ ← 7^{TH} ORDER

APP TAYLOR'S FORMULA:

$$f(\underline{x}) = \sum_{|\underline{\alpha}| \leq N} \frac{1}{\underline{\alpha}!}\, (D^{\underline{\alpha}} f)(0)\, \underline{x}^{\underline{\alpha}} + o(|\underline{x}|^N) \text{ AS } |\underline{x}| \to 0$$

$$\left(1D \quad f(x) = \sum_{k=0}^{N} \frac{1}{k!}\, f^{(k)}(0)\, x^k + o(|x|^N) \right)$$

BACK TO LAPLACE

GOAL FIND AN ASYMPTOTIC EXP FOR $I[\varepsilon] = \int_{\mathbb{R}^N} a(\underline{x})\, e^{\frac{\phi(\underline{x})}{\varepsilon}}\, d\underline{x}$

1) $a = a(\underline{x})$ SMOOTH & COMPACT SUPP

2) $\phi : \mathbb{R}^N \to \mathbb{R}$ HAS A GLOBAL MAX AT $\underline{0}$ WITH

$D\phi(\underline{0}) = (\phi_{x_1}, \cdots, \phi_{x_N})(\underline{0}) = \underline{0}$

AND $D^2\phi(\underline{0}) = \begin{bmatrix} \phi_{x_1x_1} & \cdots & \phi_{x_1x_N} \\ \vdots & & \\ \phi_{x_Nx_1} & \cdots & \phi_{x_Nx_N} \end{bmatrix}(\underline{0}) < 0$ ← ALL EIGEN-VALUES ARE < 0

SPECIAL CASE $\phi(\underline{x}) = -\frac{1}{2}\left(a_1 x_1^2 + \cdots + a_N x_N^2\right)$ WITH $a_i > 0$

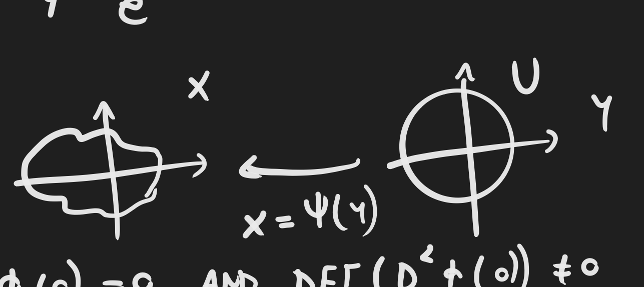
↳ QUADRATIC

$$\left(1D: \quad I[\varepsilon] \sim \sum_{k=0}^{\infty} \frac{a^{(k)}(0)}{k!}\, c_k\, \varepsilon^{\frac{k+1}{2}} \right)$$

HERE $I[\varepsilon] \sim \sum_{\underline{\alpha}} \frac{D^{\underline{\alpha}} a(0)}{\underline{\alpha}!}\, c_{\underline{\alpha}}\, \varepsilon^{\frac{|\alpha| + N}{2}}$ AS $\varepsilon \to 0$

WHERE $c_{\underline{\alpha}} =: \int_{\mathbb{R}^N} y^{\underline{\alpha}}\, e^{-\frac{1}{2}(a_1 y_1^2 + \cdots + a_N y_N^2)}\, dy$

GENERAL CASE



MORSE LEMMA IF $D\phi(0) = 0$ AND $\det(D^2\phi(0)) \neq 0$ THEN THERE IS $\psi : U \to V$ SUCH THAT

$$\phi(\psi(y)) = \phi(0) + \tfrac{1}{2}\left(y_1^2 + \cdots + y_M^2 - y_{M+1}^2 - \cdots - y_N^2\right)$$

M = # OF POSITIVE EIGENVALUES OF $D^2\phi(0)$

HERE $M = 0 \Longrightarrow \phi(\psi(y)) = \frac{1}{2}(-y_1^2 - \cdots - y_N^2) = -1/2\,(y_1^2 + \cdots + y_N^2)$ ← PARABOLOID

GENERAL LAPLACE $I[\varepsilon] \sim \sum_{k=0}^{\infty} (L_{2k}a)(x_0)\, e^{\frac{\phi(x_0)}{\varepsilon}}\, \varepsilon^{\frac{k+N}{2}}$

$$= \frac{(2\pi\varepsilon)^{N/2}}{\sqrt{|\det D^2\phi(x_0)|}}\, a(x_0)\, e^{\frac{\phi(x_0)}{\varepsilon}} + o(\sqrt{\varepsilon})$$