Homework 5 Solutions

1. The roots are (from the order the terms are listed):

   - $r = 0$, repeated twice
   - $r = -4$, repeated three times
   - $r = -7$
   - $r = \pm 3i$, repeated three times $(r^2 + 9 = 0 \Rightarrow r = \sqrt{-9})$
   - $r = -1 \pm 3i$, repeated twice $(r^2 + 2r + 10 = 0$, use quadratic formula$)$

   We can use this to compose the general solution:

$$y(t) = A + Bt + Ce^{-4t} + Dte^{-4t} + Et^2e^{-4t} + Fe^{-7t}$$

$$+G\cos(3t) + H\sin(3t) + It\cos(3t) + Jt\sin(3t) + Kt^2\cos(3t) + Lt^2\sin(3t)$$

$$+Me^{-t}\cos(3t) + Ne^{-t}\sin(3t) + Oe^{-t}t\cos(3t) + Pe^{-t}t\sin(3t)$$

2. Auxiliary equation: $r^2 = \lambda$

   Case 1: $\lambda > 0$

   Because λ is positive,
$$r^2 = \lambda \Longrightarrow \pm\sqrt{\lambda}.$$
   If we let $\omega = \sqrt{\lambda}$, then
$$r = \pm\omega.$$
   This gives the general solution
$$y(t) = Ae^{\omega t} + Be^{-\omega t}.$$
   Using the initial condition,
$$y(3) = 0 \Longrightarrow Ae^{3\omega} + Be^{-3\omega} = 0$$
$$\Longrightarrow B = -Ae^{6\omega}$$
   so
$$y(t) = Ae^{\omega t} - Ae^{6\omega}e^{-\omega t}$$
$$\Longrightarrow y'(t) = A\omega e^{\omega t} + Ae^{6\omega}\omega e^{-\omega t}$$
   Using the other initial condition,
$$y'(0) = 0 \Longrightarrow A\omega e^0 + Ae^{6\omega}\omega e^0 = 0$$
$$\Longrightarrow A\omega + Ae^{6\omega}\omega = 0$$

$$\Longrightarrow A\omega(1+e^{6\omega}) = 0.$$

Because $(1+e^{6\omega})$ must be positive, this implies either that $\omega = 0$ or that $A = 0$. The former contradicts our assumption that $\lambda > 0$, because $\omega = \sqrt{\lambda} > 0$. Therefore, we must have $A = 0$, and so our solution becomes $y(t) = 0$. Thus we have no nonzero solutions in the $\lambda > 0$ case.

<u>Case 2: $\lambda = 0$</u> In this case, the auxiliary equation $r^2 = 0$ has a repeated root at 0, so the general solution is

$$y(t) = A + Bt.$$

Using the initial conditions,

$$y(3) = 0 \Longrightarrow 0 = A + 3B$$

$$\Longrightarrow A = -3B$$

$$\Longrightarrow y(t) = -3B + Bt$$

$$\Longrightarrow y'(t) = B$$

$$y'(0) = 0 \Longrightarrow B = 0,$$

so we have $y(t) = 0$ and thus there are no nozero solutions in this case.

<u>Case 3: $\lambda < 0$</u>

Because $\lambda < 0$, $r = \pm\sqrt{\lambda}$ is imaginary, so there exists some positive real ω such that $r = \pm\omega i$. Our general solution is then

$$y(t) + A\cos(\omega t) + B\sin(\omega t).$$

Using the initial condition,

$$y(3) = 0 \Longrightarrow A\cos(3\omega) + B\sin(3\omega) = 0$$

$$\Longrightarrow A\cos(3\omega) = -B\sin(3\omega).$$

We also have that

$$y'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t),$$

so using the other initial condition,

$$y'(0) = 0 \Longrightarrow -A\omega\sin(0) + B\omega\cos(0) = 0$$

$$\Longrightarrow B\omega = 0,$$

and because $\omega \neq 0$ this implies that $B = 0$. Substituting this into our earlier equation, we have that

$$A\cos(3\omega) = 0$$

$$\Longrightarrow \cos(3\omega) = 0.$$

Cosine vanishes for arguments $n\pi/2$ where n is odd, equivalently when $n = 2m + 1$ for $m \in \{0, 1, 2, \dots\}$. Therefore,

$$3\omega = (2m+1)\frac{\pi}{2}$$
$$\implies \omega = (2m+1)\frac{\pi}{6}$$
$$\implies \lambda = -\omega^2 = -\left((2m+1)\frac{\pi}{6}\right)^2.$$

Finally, we have that the eigenvalues are

$$\lambda = -\left((2m+1)\frac{\pi}{6}\right)^2$$

with corresponding eigenfunctions

$$y = \cos(\omega t) = \cos\left((2m+1)\frac{\pi}{6}t\right),$$

for $m \in \{0, 1, 2, \dots\}$.

3. First solve the complementary solution:

$$r^2 = 5r + 4 = 0$$
$$\implies r = 4, 1$$
$$\implies y_c = C_1 e^{4t} + C_2 e^t.$$

For the particular solution, we guess the form:

$$y_p = A\cos(2t) + B\sin(2t).$$

Plugging this into the ODE:

$$y_p'' - 5y_p' + 4y_p = 20\cos(2t) + 30\sin(t)$$
$$\implies -4y_p + 10A\sin(2t) - 10B\cos(2t) + 4y_p = 20\cos(2t) + 30\sin(2t).$$
$$\implies 10A\sin(2t) - 10B\cos(2t) = 20\cos(2t) + 30\sin(2t).$$

For this to match, we must have $A = 3$ and $B = -2$, so

$$y_p = 3\cos(2t) - 2\sin(2t).$$

Our general solution is

$$y(t) = y_c + y_p$$
$$= C_1 e^{4t} + C_2 e^t + 3\cos(2t) - 2\sin(2t).$$

Using the initial condition $y(0) = 1$, we have

$$C_1 + C_2 + 3 = 1$$

$$\implies C_1 = -2 - C_2. \tag{1}$$

Using the other initial condition $y'(0) = 3$, we have

$$4C_1e^0 + C_2e^0 - 6\sin(0) - 4\cos(0) = 3$$

$$\implies 4C_1 + C_2 - 4 = 3,$$

and substituting (1) into this, we get

$$4(-2 - C_2) + C_2 - 4 = 3$$

$$\implies -8 - 3C_2 - 4 = 3$$

$$\implies C_2 = -5,$$

and substituting this into the earlier equation we have $C_1 = 3$. Therefore, our solution is

$$y(t) = 3e^{4t} - 5e^t + 3\cos(2t) - 2\sin(2t).$$

4. (a) For the homogeneous solutions, the auxiliary equation is

$$r^2 - 3r + 2 = 0$$

$$\implies r = 1, 2$$

So we can't guess Ae^t, instead we guess Ate^t.

(b) First let us check if e^{2t} coincides with the homogeneous solution:

$$r^2 - 3r + 2 = 0$$

$$\implies r = 1, 2$$

It dues coincide, so we need to multiply our guess by t:

$$y_p = (e^{2t}t)(At^2 + Bt + C)$$

(c)

$$y_p = A\sin(2t) + B\cos(2t)$$

(d) Check if it does or does not coincide:

$$r^2 - 2r + 5 = 0$$

$$\implies r = 1 \pm 2i$$

This does coindcide, so we guess

$$y_p = Ate^t\cos(2t) + Bte^t\sin(2t)$$

(e)

5. (a)

$$(D^2 - 4D + 4I)(y) = e^t$$
$$\implies (D-2I)(D-2I)(y) = E^t.$$

If we let $z = (D-2I)(y)$, then we have

$$(D-2I)(z) = e^t$$
$$\implies z' - 2z = e^t.$$

We can use integrating factors to solve this. The integrating factor is

$$e^{\int -2td} = e^{-2t},$$

so we get

$$e^{-2t}z' - e^{-2t}2z = (e^{-2t}z)' = e^{-2t}e^t = e^{-t}.$$

Integrating both sides,

$$e^{-2t}z = -e^{-t} + C_1$$
$$z = \frac{-e^{-t}}{e^{-2t}} + \frac{C_1}{e^{-2t}}$$
$$= -e^t + C_1 e^{2t}.$$

But $z = (D-2I)(y) = y' - 2y$, so

$$y' - 2y = -e^t + C_1 e^{2t}.$$

Using the same integrating factor e^{-2t},

$$e^{-2t}y' - 2e^{-2t}y = (e^{-2t}y) = -e^{-t} + C_1,$$

and by integrating both sides we get

$$e^{-2t}y = e^{-t} + C_1 t + C_2$$
$$\implies y = e^t + C_1 te^{2t} + C_2 e^{2t}.$$

(b)

$$(D^2 - 5D + 6I)(y) = e^{3t}$$
$$\implies (D-3)(D-2)(y) = e^{3t}.$$

If we let $z = (D-2)(y)$, then we have

$$(D-3)(z) = e^{3t}$$
$$\implies z' - 3z = e^{3t}.$$

We can solve this with integrating factors:

$$e^{\int -3dt} = e^{-3t}$$

$$\Longrightarrow (e^{-3t}z)' = 1.$$

Integrating both sides gives us

$$e^{-2t}z = t + C_1$$

$$\Longrightarrow z = e^{3t}t + C_1e^{3t}.$$

But $z = (D-2)(y) = y' - 2y$, so we have

$$y' - 2y = e^{3t}t + C_1e^{3t}.$$

Again using integrating factors:

$$e^{\int -2dt} = e^{-2t}$$

$$\Longrightarrow (e^{-2t}y)' = e^t t + C_1e^t$$

and integrating both sides:

$$e^{-2t}y = (t-1)e^t + C_1e^t + C_2$$

$$\Longrightarrow y = e^{3t}(t-1) + C_1e^{3t} + C_2e^{2t}.$$

6. We can set up this problem as:

$$\begin{cases} 2y'' + 2y' + y = 0 \\ y(0) = 5 \\ y'(0) = 3 \end{cases}$$

The term $2y''$ is $F = ma$, the $2y'$ term is damping, and the coefficient of 1 on the y term is due to the spring constant of 1. The auxiliary equation is

$$2r^2 + 2r + 1 = 0.$$

Using the quadratic formula, we get

$$r = \frac{-2 \pm 2i}{4}$$

$$\Longrightarrow r = -\frac{1}{2} \pm \frac{1}{2}i,$$

which gives us the solution:

$$y(t) = Ae^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) + Be^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right).$$

Using the position initial conditions:

$$y(0) = 5$$

$$\implies A = 5.$$

Tp use the velocity initial condition, we must take the derivative of our solution:

$$y'(t) = -\frac{1}{2}Ae^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right) - \frac{1}{2}Ae^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right)$$

$$+\frac{1}{2}Be^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) - \frac{1}{2}Be^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right)$$

$$y'(0) = 0 \implies -\frac{1}{2}A + \frac{1}{2}B = 0$$

$$\implies -\frac{5}{2} + \frac{1}{2}B = 3$$

$$\implies B = 11.$$

Therefore, our solution is

$$y(t) = 5e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) + 11e^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right).$$

We see that this is an oscillating but decaying solution. The damping is an energy sink in our system, and without a force term, damping will take our system into an intertial state.