APMA 0359 - Homework 6 Solutions

October 22, 2023

1. Use variation of parameters to find the general solution of the following ODE. Simplify your answers.

   (a) $t^2y'' + 4t^2y' + 4t^2y = e^{-2t}$

Solution: First, we put the ODE into standard form:

$$y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}$$

Next, we find the homogenous solution:

$$y'' + ty' + 4y = 0$$

Our characteristic equation is

$$r^2 + 4y + 4 = 0 \implies (r+2)^2 = 0$$

Thus, we have

$$y_0 = Ae^{-2t} + Bte^{-2t}.$$

We guess y_p will be of the same form,

$$y_p = u(t)e^{-2t} + v(t)te^{-2t}.$$

Thus, using the Wronskian we get the new set of equations

$$\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2e^{-2t}t \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{-2t}}{t^2} \end{bmatrix}.$$

Solving with Cramer's rule, we get

$$u'(t) = \frac{\begin{bmatrix} 0 & te^{-2t} \\ \frac{e^{-2t}}{t^2} & e^{-2t} - 2e^{-2t}t \end{bmatrix}}{\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2e^{-2t}t \end{bmatrix}} = \frac{0 - (te^{-2t})(\frac{e^{-2t}}{t^2})}{(e^{-2t})(e^{-2t} - 2e^{-2t}t) - (-2e^{-2t}te^{-2t})} = \frac{e^{-4t}/t}{e^{-4t}} = \frac{-1}{t}$$

$$v'(t) = \frac{\begin{bmatrix} e^{-2t} & 0 \\ -2e^{-2t} & \frac{e^{-2t}}{t^2} \end{bmatrix}}{\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2e^{-2t}t \end{bmatrix}} = \frac{e^{-2t}(\frac{e^{-2t}t^2}{)} - 0}{e^{-4t}} = \frac{e^{-4t}/t^2}{e^{-4t}} = \frac{1}{t^2}.$$

Solving for $u(t)$ and $v(t)$, we find

$$u(t) = -\int t^{-1} dt = -\ln|t|$$

and

$$v(t) = \int t^{-2} dt = -\frac{1}{t}.$$

Thus, we have that

$$y(t) = y_0 + y_p = Ae^{-2t} + Bte^{-2t} + \ln|t|e^{-2t} - \frac{te^{-2t}}{t} = De^{-2t} + Bte^{-2t} - \ln|t|e^{-2t}.$$

(b) $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$

Solution: We first put the ODE into the standard form,

$$y'' - \frac{(t+2)}{t}y' + \frac{(t+2)}{t^2}y = 2t.$$

We are assuming t and te^t are solutions to the homogenou sequations. THus,

$$y_0 = At + Bte^t.$$

We guess the same form for y_p, where

$$y_p = u(t)t + v(t)te^t.$$

We get the system of equations

$$\begin{bmatrix} t & te^t \\ 1 & e^t + te^t \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \end{bmatrix}.$$

Using Cramer's rule, we obtain

$$u'(t) = \frac{\begin{bmatrix} 0 & te^t \\ 2t & e^t + te^t \end{bmatrix}}{\begin{bmatrix} t & te^t \\ 1 & e^t + te^t \end{bmatrix}} = \frac{0 - 2t^2e^t}{te^t + t^2e^t - te^t} = \frac{-2t^2e^t}{t^2e^t} = -2.$$

and

$$v'(t) = \frac{\begin{bmatrix} t & 0 \\ 1 & 2t \end{bmatrix}}{\begin{bmatrix} t & te^t \\ 1 & e^t + te^t \end{bmatrix}} = \frac{2t^2 - 0}{te^t + t^2e^t - te^t} = \frac{2t^2}{t^2e^t} = \frac{2}{e^t} = 2e^t.$$

Thus, we solve to find

$$u(t) = \int -2dt = -2t$$

and

$$v(t) = 2\int e^{-t}dt = -2e^{-t}.$$

Thus, we have that

$$y(t) = y_0 + y_p = At + Bte^t - 2t^2 - 2t = Dt + Bte^t - 2t^2.$$

2. Find the general solution of $y'' + y = \cos(t)$

(a) Using undetermined coefficients

Solution: We first solve for the homogenous equation,

$$y'' + y = \cos(t).$$

Thus, we have that

$$r^2 + 1 = 0 \implies r = \pm i$$

and we get the homogenous solution

$$y_0 = A\cos(t) + B\sin(t).$$

Since we have $\cos(t)$ in the homogenous, we must write our particular solution as

$$y_p = At\cos(t) + Bt\sin(t).$$

We then find

$$\begin{aligned} y_p' &= A(\cos(t) - t\sin(t)) + B(\sin(t) + t\cos(t)) \\ &= A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t). \end{aligned}$$

and

$$\begin{aligned} y_p'' &= -A\sin(t) - A(\sin(t) + t\cos(t)) + B\cos(t) + B(\cos(t) - t\sin(t)) \\ &= -A\sin(t) - A\sin(t) - At\cos(t) + B\cos(t) + B\cos(t) - Bt\sin(t) \\ &= 2B\cos(t) - 2A\sin(t) - At\cos(t) - Bt\sin(t). \end{aligned}$$

Using $y_p'' + y_p = \cos(t)$, we have

$$2B\cos(t) - 2A\sin(t) = \cos(t) \implies B = \frac{1}{2}, A = 0.$$

Therefore, we arrive at the solution

$$y(t) = A\cos(t) + B\sin(t) + \frac{t}{2}\sin(t).$$

(b) Using variation of parameters

Solution: We obtain the same homogenous equation as above,

$$y_0 = \cos(t) + B\sin(t).$$

Thus, we also have that

$$y_p = u(t)\cos(t) + v(t)\sin(t).$$

Thus, we get the system of equations

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}.$$

Using Cramer's rule, we obtain

$$u'(t) = \frac{\begin{bmatrix} 0 & \sin(t) \\ \cos(t) & \cos(t) \end{bmatrix}}{\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}} = \frac{0 - \sin(t)\cos(t)}{\cos^2(t) + \sin^2(t)} = \frac{-\sin(t)\cos(t)}{1} = -\sin(t)\cos(t)$$

$$v'(t) = \frac{\begin{bmatrix} \cos(t) & 0 \\ -\sin(t) & \cos(t) \end{bmatrix}}{\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}} = \frac{\cos^2(t) - 0}{1} = \cos^2(t)$$

We solve

$$u(t) = -\int \sin(t)\cos(t)dt.$$

By letting $u = \cos(t)$ and $du = -\sin(t)dt$, we find

$$\int udu = \frac{u^2}{2} \implies u(t) = \frac{\cos^2(t)}{2}.$$

Next, we solve for

$$\begin{aligned} v(t) &= \int \cos^2(t)dt \\ &= \frac{1}{2}\cos(2t) + \frac{1}{2}dt \\ &= \frac{1}{2}\cos(2t)dt + \int \frac{1}{2}dt \\ &= \frac{\sin(2t)}{4} + \frac{t}{2} \\ &= \frac{1}{2}(t + \sin(t)\cos(t)). \end{aligned}$$

Thus, using $\cos^2(t) = 1 - \sin^2(t)$, we have

$$y_p = \frac{\cos^3(t)}{2} + \frac{1}{2}(t + \sin(t)\cos(t))\sin(t) = \frac{1}{2}(\cos(t) + t\sin(t))$$

Therefore,

$$y = A\cos(t) + B\sin(t) + \frac{\cos(t)}{2} + \frac{t\sin(t)}{2} = D\cos(t) + B\sin(t) + \frac{t\sin(t)}{2}$$

and our solutions match.

3. In this problem, we'll rederive the Var of Par equtions, but for

$$y'' - 5y' + 6y = f(t)$$

where $f(t)$ is an inhomogenous term. Recall that the homogenous solution is $y_0 = Ae^{2t} + Be^{3t}$.

Variation of Parameters: Suppose y_p is of the form

$$y_p = u(t)e^{2t} + v(t)e^{3t}$$

and suppose for simplicity that

$$e^{2t}u'(t) + e^{3t}v'(t) = 0$$

Calculate $(y_p)'$ and $(y_p)''$ and plug into the ODE to show

$$(2e^{2t})u'(t) + (3e^{3t})v'(t) = f(t)$$

Solution: We first find

$$\begin{aligned} y_p' =& \frac{d}{dt}(u(t)e^{2t} + v(t)e^{3t}) \\ =& u'(t)e^{2t} + 2e^{2t}u(t) + v'(t)e^{3t} + 2e^{3t}v(t) \\ =& 2u(t)e^{2t} + 3v(t)e^{3t}. \end{aligned}$$

Next, we find

$$\begin{aligned} y_p'' = \frac{d}{dt}(y_p') =& 2(u'(t)e^{2t} + 2e^{2t}u(t)) + 3(v'(t)e^{3t} + 3e^{3t}v(t)) \\ =& 2u'(t)e^{2t} + 4e^{2t} + 4e^{2t}u(t) + 3v'(t)e^{3t} + 9e^{3t}v(t). \end{aligned}$$

Plugging into $y'' - 5y' + 6y = f(t)$, we find

$$\begin{aligned} f(t) = 2u'(t)e^{2t} &+ 4e^{2t}u(t) + 3v'(t)e^{3t} + 9e^{3t}v(t) \\ &-10e^{2t}u(t) \qquad - 15e^{2t}v(t) \\ &+6u(t)e^{2t} \qquad + 6e^{3t}v(t). \end{aligned}$$

Thus, we found that

$$f(t) = 2u'(t)e^{2t} + 3v'(t)e^{3t}.$$

4. Use tabular integration to find $\mathcal{L}\{t^2 2\}$

Solution: We apply tabular integration by writing

	Differentiate		Integrate
+	t^2	↘	e^{-st}
-	$2t$	↘	$-e^{-st}/s$
+	2	↘	e^{-st}/s^2
-	0	↘	$-e^{-st}/s^3$

Thus, we have

$$\mathcal{L}(t^2) = \int_0^\infty t^2 e^{-st} dt = \left[-\frac{t^2 e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right]_0^\infty = \frac{2}{s^3}.$$

5. Use complex exponentials to find

$$\mathcal{L}\{\cos(3t)\} \text{ and } \mathcal{L}\{\sin(3t)\}$$

Solution: Euler's formula tells us that $e^{-x} = \cos(x) + i\sin(x)$.
Thus,

$$\cos(3t) = \frac{e^{3it} + e^{-3it}}{2} \text{ and } \sin(3t) = \frac{e^{3it} - e^{-3it}}{2i}.$$

We calculate the transforms as

$$\begin{aligned}\mathcal{L}(\cos(3t)) =& \mathcal{L}(\frac{e^{3it} + e^{-3it}}{2}) \\ =& \frac{1}{2}\int_0^\infty (e^{(3i-s)t} + e^{-3i-s)t})dt \\ =& \frac{1}{2}\left[\frac{e^{(3i-s)t}}{3i-s} + \frac{e^{(-3i-s)t}}{-3i-s}]\right]_0^\infty \\ =& \frac{s}{s^2+9}\end{aligned}$$

and similarly,

$$\begin{aligned}\mathcal{L}(\sin(3t)) =& \frac{1}{2i}\int_0^\infty (e^{(3i-s)t} + e^{(-3i-s)t}) \\ =& \frac{3}{s^2+9}\end{aligned}$$

6. Find examples of functions $f(t)$ and $g(t)$ with

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

Most guesses should work, but try out constant/exponential functions.

Solution: Following the hint, we use $f(t) = t$ and $g(t) = c$, where c is a constant. Then, we find

$$\mathcal{L}(f(t)g(t)) = \mathcal{L}(ct) = \frac{c}{s^2} \neq \frac{c}{s^3} = \frac{1}{s^2}\frac{c}{s} = \mathcal{L}(t)\mathcal{L}(c) = \mathcal{L}(f(t))\mathcal{L}(g(t)).$$

7. Use Laplace Transforms to solve

$$\begin{cases} y'' + 9y = \cos(2t) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solution: We take the transform of the equation to find

$$\mathcal{L}(y'') + 9\mathcal{L}(y) = 2(\cos(2t)).$$

We call $\mathcal{L} = Y(s)$.
Then, we get

$$s^2Y(s) - sy(0) + 9Y(s) = \frac{s}{s^2+4}$$

Factoring and using the initial condition, we get

$$(s^2+9)Y(s) - s = \frac{s}{s^2+4}.$$

Thus, we see that

$$Y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{(s^2+9)} = \frac{s^3+5s}{(s^2+4)(s^2+9)}.$$

We use partial fraction decomposition and find

$$\begin{aligned} Y(s) =& \frac{(As+B)}{s^2+4} + \frac{(cs+D)}{s^2+9} \\ =& \frac{As^3+9As+Bs^2+9B+Cs^3+4Cs+Ds^2+4D}{(s^2+4)(s^2+9)}. \end{aligned}$$

Setting the numerators equal, we find

$$B = 0, D = 0, A + C = 1 \text{ and } 9A + 4C = 5 \implies C = -4/5, A = 1/5.$$

Thus,

$$Y(s) = \frac{s}{5(s^2+4)} + \frac{4s}{5(s^2+9)}.$$

So taking the inverse of the Laplace transform, we find

$$y(t) = \frac{1}{5}\cos(2t) + \frac{4}{5}\cos(3t).$$