LECTURE: COMPLEX ROOTS

1. Quick Facts about Complex Numbers

Definition:

$$i = \sqrt{-1}$$

This implies that $i^2 = -1$

From this you can create **complex numbers** like $2 + 3i$

Definition: (Real and Imaginary Parts)

$$\text{Re}\,(2 + 3i) = 2 \qquad \text{Im}\,(2 + 3i) = 3$$

We can generalize exponential functions to include complex numbers:

Definition:

$$e^{it} = \cos(t) + i\sin(t)$$

Example 1:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1$$

This gives what some people call the most beautiful formula in math:

$$e^{i\pi} + 1 = 0$$

Notice it relates the 5 most important constants of math: $0, 1, e, \pi, i$ ☺

2. Complex Roots

Video: Complex Roots

Example 2:

$$y'' + y = 0$$

$$\textbf{Aux:} \quad r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm\sqrt{-1} \Rightarrow r = \pm i$$

This means that:

$$y = Ae^{it} + Be^{-it}$$

Question: How to get real solutions from this?

$$\begin{aligned} y =& A\left(\cos(t) + i\sin(t)\right) + B\left(\cos(-t) + i\sin(-t)\right) \\ =& A\cos(t) + iA\sin(t) + B\cos(t) - iB\sin(t) \\ =& \underbrace{(A+B)}_{A}\cos(t) + \underbrace{i\,(A-B)}_{B}\sin(t) \end{aligned}$$

Since A and B are arbitrary, so are $A+B$ and $i(A-B)$ and so

$$y = A\cos(t) + B\sin(t)$$

Note: Here A and B are complex coefficients, which is not a big problem because in our examples below we will get real constants.

But if you really want to get real solutions you can proceed as follows:

First write $A = A_1 + iA_2$ and $B = B_1 + iB_2$ and split up into real and imaginary parts as follows:

$$\begin{aligned} y =& A\cos(t) + B\sin(t) \\ =& (A_1 + iA_2)\cos(t) + (B_1 + iB_2)\sin(t) \\ =& (A_1\cos(t) + B_1\sin(t)) + i\,(A_2\cos(t) + B_2\sin(t)) \end{aligned}$$

Now take real parts on both sides:

$$\text{Re}(y) = A_1\cos(t) + B_1\sin(t)$$

But since y is a real solution, we have $\text{Re}(y) = y$ and so we get

$$y = A_1\cos(t) + B_1\sin(t)$$

Where A_1 and B_1 are arbitrary real constants

Example 3:

$$\begin{cases} y'' + 6y' + 10y = 0 \\ y(0) = 1 \\ y'(0) = -4 \end{cases}$$

Auxiliary equation:

$$\begin{aligned} r^2 + 6r + 10 =& 0 \\ (r+3)^2 - 9 + 10 =& 0 \\ (r+3)^2 =& -1 \\ r + 3 =& \pm\sqrt{-1} = \pm i \\ r =& -3 \pm i \end{aligned}$$

Mnemonic: $e^{(-3+i)t} = e^{-3t}e^{it} = e^{-3t}\cos(t) + ie^{-3t}\sin(t)$

You could repeat the process above, or just go directly to

$$y = Ae^{-3t}\cos(t) + Be^{-3t}\sin(t)$$

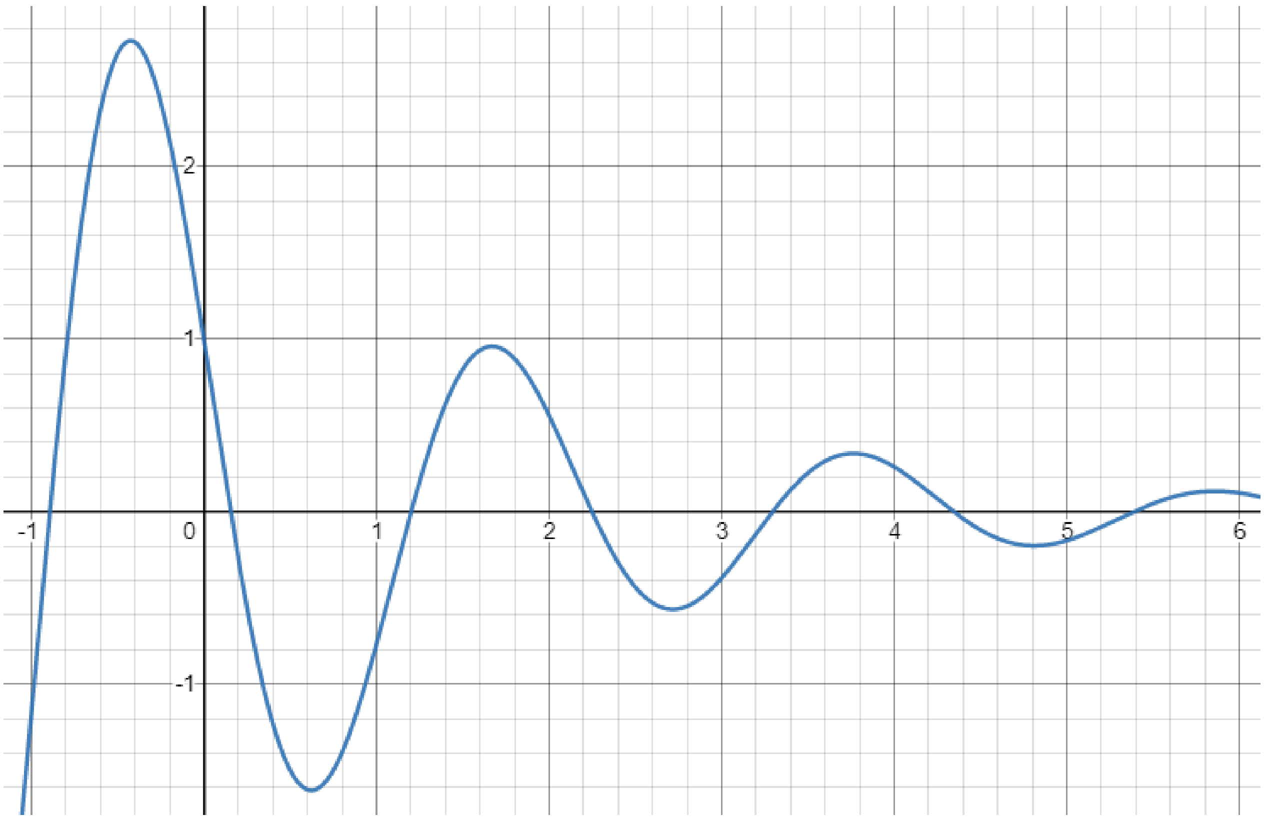
$$\begin{aligned} y(0) =& 1 \\ Ae^0\cos(0) + Be^0\sin(0) =& 1 \\ A + 0 =& 1 \\ A =& 1 \end{aligned}$$

$$y = e^{-3t}\cos(t) + Be^{-3t}\sin(t)$$

$$y'(t) = -3e^{-3t}\cos(t) + e^{-3t}(-\sin(t)) - 3Be^{-3t}\sin(t) + Be^{-3t}\cos(t)$$

$$\begin{aligned} y'(0) =& -4 \\ -3e^0\cos(0) - e^0(\sin(0)) - 3Be^0\sin(0) + Be^0\cos(0) =& -4 \\ -3 + 0 - 0 + B =& -4 \\ -3 + B =& -4 \\ B =& -4 + 3 = -1 \end{aligned}$$

$$y = e^{-3t}\cos(t) - e^{-3t}\sin(t)$$



Solution gets damped quite quickly because of the e^{-3t} term.

Example 4: (more practice)

$$\begin{cases} y'' + 4y = 0 \\ y(0) = 3 \\ y'(0) = -4 \end{cases}$$

$$\textbf{Aux:}\ r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = \pm 2i$$
$$e^{2it} = e^{(2t)i} = \cos(2t) + i\sin(2t)$$

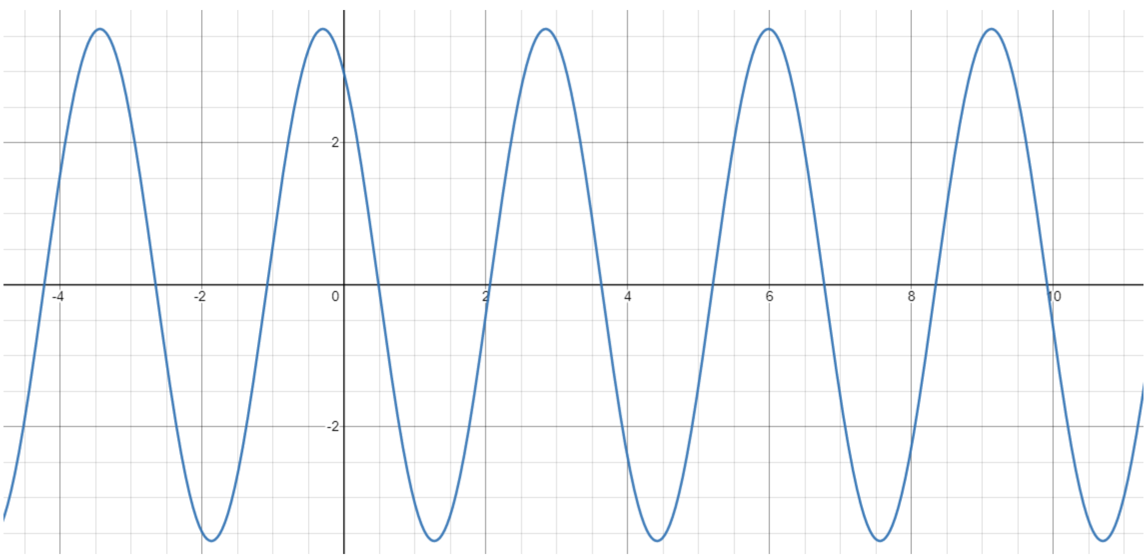
$$y = A\cos(2t) + B\sin(2t)$$

$$\begin{aligned} y(0) &= 3 \\ A\cos(0) + B\sin(0) &= 3 \\ A(1) + B(0) &= 3 \\ A &= 3 \end{aligned}$$

$$y' = (A\cos(2t) + B\sin(2t))' = -2A\sin(2t) + 2B\cos(2t)$$

$$\begin{aligned} y'(0) &= -4 \\ -2A\sin(0) + 2B\cos(0) &= -4 \\ -2A(0) + 2B(1) &= -4 \\ 2B &= -4 \\ B &= -2 \end{aligned}$$

$$y = 3\cos(2t) - 2\sin(2t)$$



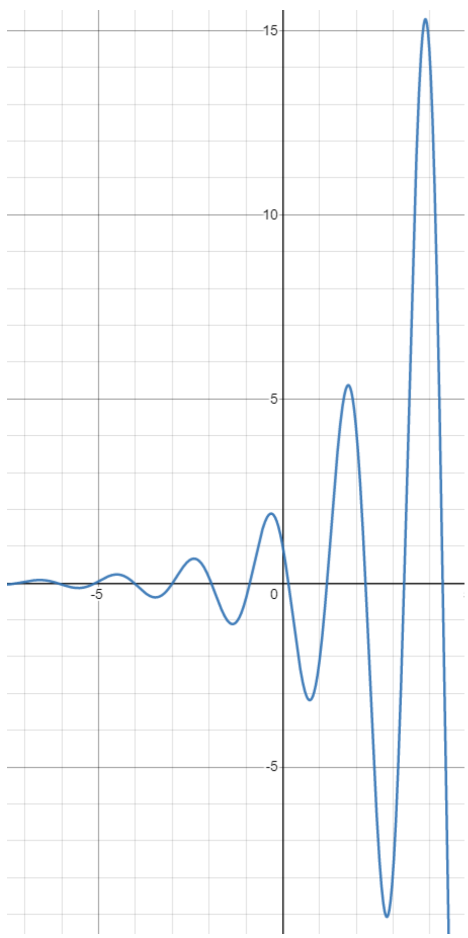
Example 5: (more practice)

$$y'' - 4y' + 13y = 0$$

$$\begin{aligned} r^2 - 4r + 13 =& 0 \\ (r-2)^2 - 4 + 13 =& 0 \\ (r-2)^2 =& -9 \\ r - 2 =& \pm\sqrt{-9} = \pm 3i \\ r =& 2 \pm 3i \end{aligned}$$

$$e^{(2+3i)t} = e^{2t}e^{3ti} = e^{2t}\cos(3t) + ie^{2t}\sin(3t)$$

$$y = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$$



The e^{2t} term causes the oscillations to blow up and the $\cos(3t)$ term makes things oscillate fast.

3. Summary

To summarize, there are 3 possible scenarios for the auxiliary equation:

Example 6: (more practice)

(a) Distinct Roots

$$y'' - 5y' + 6y = 0 \Rightarrow r^2 - 5r + 6 \Rightarrow r = 2 \text{ or } r = 3$$

$$y = Ae^{2t} + Be^{3t}$$

(b) Complex Roots

$$y'' - 4y' + 13y = 0 \Rightarrow r^2 - 4r + 13 = 0 \Rightarrow r = 2 \pm 3i$$

$$y = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$$

(c) Repeated Roots

$$y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r = 1$$

$$y = Ae^{t} + Bte^{t}$$

Note: For higher order equations, it's pretty much exactly the same:

Example 7:

Solve the ODE whose auxiliary equation is

$$r(r-1)(r-2)^3(r^2-4r+13)^2 = 0$$

$r=0$, $r=1$, $r=2$ (repeated 3 times), $r=2\pm 3i$ (repeated twice):

$$\begin{aligned} y = & \underbrace{Ae^{0t}}_{A} + Be^{t} + Ce^{2t} + Dte^{2t} + Et^{2}e^{2t} \\ & + Fe^{2t}\cos(3t) + Ge^{2t}\sin(3t) + Hte^{2t}\cos(3t) + Ite^{2t}\sin(3t) \end{aligned}$$

4. Boundary-Value Problems

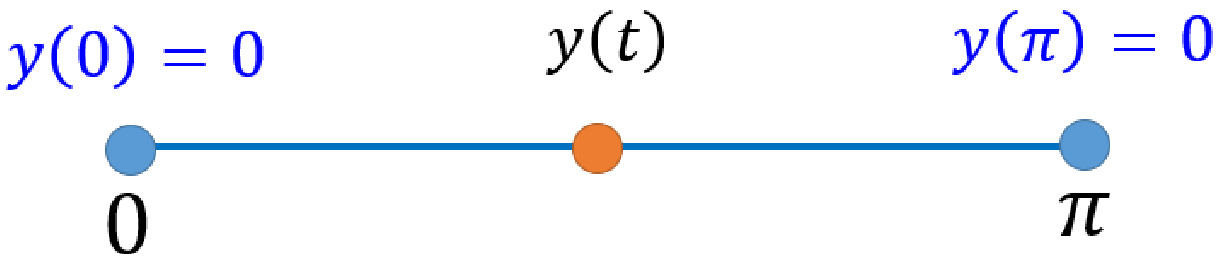
Video: Boundary-Value Problems

This is a nice application of the techniques we have learned so far, and will be super useful if you ever take PDE

Example 8:

Find the values of λ for which the ODE has nonzero solutions

$$\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$



Remarks:

(1) Before we had initial conditions like $y(0) = 3$ and $y'(0) = -4$ but now we have boundary conditions where we specify the values $y(0)$ and $y(\pi)$ at the endpoints of the interval $[0, \pi]$

(2) **Application:** Here $y(t)$ represents the temperature of a metal rod. The boundary conditions mean that we insulate the rod to have temperature 0 at the endpoints

Auxiliary Equation: $r^2 = \lambda$

The behavior of the solutions depends on the sign of λ, so it makes sense to split this into 3 cases.

Case 1: $\lambda > 0$

Notice that if $\lambda > 0$ then $\lambda = \omega^2$ for some $\omega > 0$, this avoids nasty square roots

Ex: If $\lambda = 9 = 3^2$, then $\omega = 3$

Aux: $r^2 = \lambda = \omega^2 \Rightarrow r = \pm\omega$ in which case we get

$$y = Ae^{\omega t} + Be^{-\omega t}$$

$$y(0) = Ae^0 + Be^0 = A + B = 0 \Rightarrow B = -A$$
$$y = Ae^{\omega t} - Ae^{-\omega t}$$

$$\begin{aligned} y(\pi) &= 0 \\ Ae^{\omega\pi} - Ae^{-\omega\pi} &= 0 \\ \cancel{A}e^{\omega\pi} &= \cancel{A}e^{-\omega\pi} \\ \omega\pi &= -\omega\pi \\ 2\pi\omega &= 0 \\ \omega &= 0 \end{aligned}$$

(Can cancel out A because if $A = 0$ then $y = 0$ but want $\neq 0$ solutions)

But then $\lambda = \omega^2 = 0^2 = 0$, which contradicts $\lambda > 0 \Rightarrow\Leftarrow$

Conclusion: In this case, we have no nonzero solutions

Case 2: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (repeated twice)

$$y = Ae^{0t} + Bte^{0t} = A + Bt$$
$$y(0) = A + B(0) = A = 0$$
$$y = Bt$$
$$y(\pi) = 0 \Rightarrow B\pi = 0 \Rightarrow B = 0$$

But then $y = 0t = 0 \Rightarrow\Leftarrow$ (since we want nonzero solutions)

Conclusion: In this case, we also have no nonzero solutions

Case 3: $\lambda < 0$

(To be continued next time)