LECTURE: UNDETERMINED COEFFICIENTS

1. Undetermined Coefficients

Example 1:

Find a particular solution y_p of $y'' - 5y' + 6y = 6e^{4t}$

Rule 1: If right-hand-side (RHS) is e^{rt}, guess $y_p = Ae^{rt}$

Here guess $y_p = Ae^{4t}$

$$\begin{aligned}\left(Ae^{4t}\right)'' - 5\left(Ae^{4t}\right)' + 6Ae^{4t} =& 6e^{4t}\\ 16Ae^{4t} - 5\left(4Ae^{4t}\right) + 6Ae^{4t} =& 6e^{4t}\\ 16A - 20A + 6A =& 6\\ 2A =& 6\\ A =& 3\end{aligned}$$

$$y_p = Ae^{4t} = 3e^{4t}$$

Example 2:

Find the general solution of $y'' - 2y' - 3y = 9t^2$

STEP 1: Homogeneous

$$y''-2y'-3y = 0 \Rightarrow r^2-2r-3 = 0 \Rightarrow (r-3)(r+1) = 0 \Rightarrow r = 3 \text{ or } r = -1$$

$$y_0 = Ae^{3t} + Be^{-t}$$

STEP 2: Particular

Rule 2: If the right-hand-side (RHS) is a polynomial, guess

$$y_p = \text{ polynomial of the same degree}$$

Example: If the RHS is $3t + 2$ (degree 1) then you guess $At + B$

Example: If the RHS is $2t^2 - 3t + 4$ (degree 2) you guess $At^2 + Bt + C$

Here the RHS is $9t^2$ (degree 2) so guess $y_p = At^2 + Bt + C$ (degree 2)

$$\begin{aligned}
y'' - 2y' - 3y &= 9t^2 \\
\left(At^2 + Bt + C\right)'' - 2\left(At^2 + Bt + C\right)' - 3\left(At^2 + Bt + C\right) &= 9t^2 \\
2A - 2(2At + B) - 3\left(At^2 + Bt + C\right) &= 9t^2 \\
2A - 4At - 2B - 3At^2 - 3Bt - 3C &= 9t^2 \\
-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) &= 9t^2 + 0t + 0
\end{aligned}$$

Comparing the coefficients, this gives us

$$\begin{cases} -3A = 9 \\ -4A - 3B = 0 \\ 2A - 2B - 3C = 0 \end{cases}$$

The first equation gives $A = -3$

$$-4A - 3B = 0 \Rightarrow -3B = 4A = 4(-3) = -12 \Rightarrow B = 4$$

$$-3C = -2A + 2B = -2(-3) + 2(4) = 14 \Rightarrow C = -\frac{14}{3}$$

$$y_p = At^2 + Bt + C = -3t^2 + 4t - \frac{14}{3}$$

STEP 3: General

$$y = y_0 + y_p = Ae^{3t} + Be^{-t} - 3t^2 + 4t - \frac{14}{3}$$

Example 3:

$$\begin{cases} y'' - y' = \cos(2t) \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$$

STEP 1: Homogeneous

$$r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1$$

$$y_0 = Ae^{0t} + Be^{t} = A + Be^{t}$$

STEP 2: Particular

Rule 3: cos always goes with sin, and vice-versa

Here RHS $= \cos(2t)$, so guess

$$y_p = A\cos(2t) + B\sin(2t)$$

$$\begin{aligned} y'' - y' &= \cos(2t) \\ (A\cos(2t) + B\sin(2t))'' - (A\cos(2t) + B\sin(2t))' &= \cos(2t) \\ (-4A\cos(2t) - 4B\sin(2t)) - (-2A\sin(2t) + 2B\cos(2t)) &= \cos(2t) \\ -4A\cos(2t) - 4B\sin(2t) + 2A\sin(2t) - 2B\cos(2t) &= \cos(2t) \\ (-4A - 2B)\cos(2t) + (2A - 4B)\sin(2t) &= 1\cos(2t) + 0\sin(2t) \end{aligned}$$

$$\begin{cases} -4A - 2B = 1 \\ 2A - 4B = 0 \end{cases}$$

The second equation gives $2A = 4B \Rightarrow A = 2B$

$$-4A - 2B = 1 \Rightarrow -4(2B) - 2B = 1 \Rightarrow -10B = 1 \Rightarrow B = -\frac{1}{10}$$

Hence $A = 2B = 2\left(-\frac{1}{10}\right) = -\frac{1}{5}$ and

$$y_p = -\frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t)$$

STEP 3: General

$$y = A + Be^t - \frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t)$$

STEP 4: Initial Condition

$$\begin{aligned} y(0) &= 2 \\ A + Be^0 - \frac{1}{5}\cos(0) - \frac{1}{10}\sin(0) &= 2 \\ A + B - \frac{1}{5} &= 2 \\ A + B &= 2 + \frac{1}{5} = \frac{11}{5} \end{aligned}$$

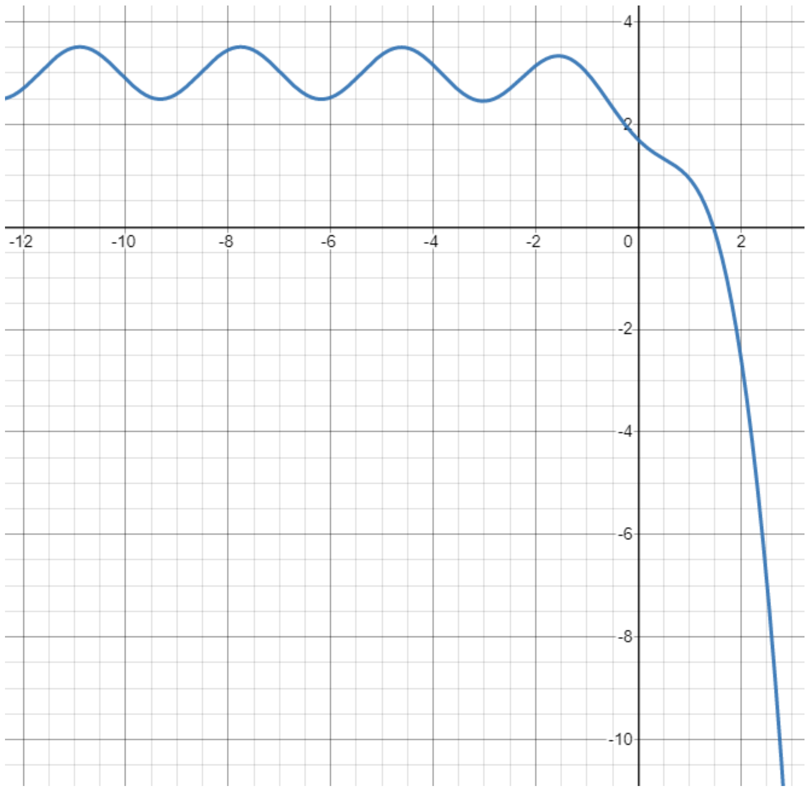
$$y' = Be^t - \frac{1}{5}(-2\sin(2t)) - \left(\frac{1}{10}\right) 2\cos(2t) = Be^t + \frac{2}{5}\sin(2t) - \frac{1}{5}\cos(2t)$$

$$y'(0) = -1 \Rightarrow Be^0 + \frac{2}{5}\sin(0) - \frac{1}{5}\cos(0) = -1 \Rightarrow B - \frac{1}{5} = -1 \Rightarrow B = -1 + \frac{1}{5} = -\frac{4}{5}$$

$$A + B = \frac{11}{5} \Rightarrow A = \frac{11}{5} - B = \frac{11}{5} + \frac{4}{5} = \frac{15}{5} = 3$$

STEP 5: Answer

$$y = 3 - \frac{4}{5}e^t - \frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t)$$



Aside: If you're not a fan of this, check out variation of parameters (next lecture) or this video, based on the factoring method:

Video: Cool Inhomogeneous Equations

2. Sums of Terms

Example 4:

Find a particular solution y_p of

$$y'' + 4y = 7e^{3t} + 2t$$

For sums of terms, best to guess the particular solutions separately:

STEP 1: For $7e^{3t}$ guess $y_p = Ae^{3t}$

$$\begin{aligned}(y_p)'' + 4y_p =& 7e^{3t} \\ \left(Ae^{3t}\right)'' + 4\left(Ae^{3t}\right) =& 7e^{3t} \\ 9Ae^{3t} + 4Ae^{3t} =& 7e^{3t} \\ 13A =& 7 \\ A =& \frac{7}{13}\end{aligned}$$

This gives $\frac{7}{13}e^{3t}$

STEP 2: For $2t$ guess $y_p = At + B$

$$\begin{aligned}(y_p)'' + 4y_p =& 2t \\ (At+B)'' + 4(At+B) =& 2t \\ 0 + 4At + 4B =& 2t \\ (4A)t + 4B =& 2t\end{aligned}$$

That is $4A = 2$ and $4B = 0$ so $A = 1/2$ and $B = 0$

This gives $\frac{t}{2}$

STEP 3: Add them up to get

$$y_p = \frac{7}{13}e^{3t} + \frac{t}{2}$$

3. Resonance

Here is an important **exception** to undetermined coefficients:

Example 5:

$$y'' - 5y' + 6y = e^{2t}$$

Here the RHS e^{2t} corresponds to the root $r = 2$

STEP 1: Homogeneous Solution

Aux: $r^2 - 5r + 6 = 0 \Rightarrow r = 2$ or $r = 3$

$$y_0 = Ae^{2t} + Be^{3t}$$

Notice: The root $r = 2$ from the RHS *coincides* with one of the roots of the auxiliary equation, so there is **resonance** (see below)

STEP 2: Particular Solution

CAN'T guess $y_p = Ae^{2t}$ since Ae^{2t} is part of the homogeneous solution! Plugging y_p in the ODE would give you 0 and not e^{2t} (try it out!)

To fix this, guess $y_p = Ate^{2t}$

$$\begin{aligned}(y_p)' &= Ae^{2t} + 2Ate^{2t} \\ (y_p)'' &= \left(Ae^{2t} + 2Ate^{2t}\right)' \\ &= 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} \\ &= 4Ae^{2t} + 4Ate^{2t}\end{aligned}$$

$$\begin{aligned} (y_p)'' - 5(y_p)' + 6(y_p) &= e^{2t} \\ \left(4Ae^{2t} + 4Ate^{2t}\right) - 5\left(Ae^{2t} + 2Ate^{2t}\right) + 6\left(Ate^{2t}\right) &= e^{2t} \\ 4Ae^{2t} + \cancel{4Ate^{2t}} - 5Ae^{2t} \cancel{-10Ate^{2t}} + \cancel{6Ate^{2t}} &= e^{2t} \\ -Ae^{2t} &= e^{2t} \\ A &= -1 \end{aligned}$$

$$y_p = Ate^{2t} = -te^{2t}$$

STEP 3: General Solution

$$y = Ae^{2t} + Be^{3t} - te^{2t}$$

Note: This "exception" is called resonance, there will be a cool application of this in a future lecture

Summary:

If the RHS $e^{2t} \rightsquigarrow r = 2$ coincides with a root $r = 2$ of the hom. equation, then there is resonance, and you guess Ate^{2t}

4. Who's that particular solution?

Example 6:

Guess the form of the particular solution

(a) $y'' + 3y' - 4y = e^{2t}$

$$\textbf{Aux: } r^2 + 3r - 4 = 0 \Rightarrow (r-1)(r+4) = 0 \Rightarrow r = 1 \text{ or } r = -4$$

$$y_0 = Ae^{t} + Be^{-4t}$$

$e^{2t} \rightsquigarrow r = 2$ which does not coincide, so

$$y_p = Ae^{2t}$$

(b) $y'' + 3y' - 4y = e^t$

$e^t \rightsquigarrow r = 1$ which coincides, so $y_p = Ate^t$ (resonance)

(c) $y'' + 3y' - 4y = t^3 e^{-4t}$

$e^{-4t} \rightsquigarrow r = -4$ which coincides, so

$$y_p = t(At^3 + Bt^2 + Ct + D)e^{-4t}$$

(d) $y'' + 3y' - 4y = e^{-4t}\cos(t)$

You have to look at $e^{-4t}\cos(t)$ as a *whole*. It corresponds to $r = -4 \pm i$ which does **not** coincide, so

$$y_p = Ae^{-4t}\cos(t) + Be^{-4t}\sin(t)$$

(e) $y'' - 4y' + 13y = e^{2t}$

$r^2 - 4r + 13 = 0 \Rightarrow r = 2 \pm 3i$

$$y_0 = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$$

$e^{2t} \rightsquigarrow r = 2$ which does not coincide, so $y_p = Ae^{2t}$

(f) $y'' - 4y' + 13y = e^{2t}\sin(3t)$

$e^{2t}\sin(3t) \rightsquigarrow r = 2 \pm 3i$ which coincides, so

$$y_p = Ate^{2t}\cos(3t) + Bte^{2t}\sin(3t)$$

(g) $y'' - 4y' + 4y = e^{2t}$

Aux: $r^2 - 4r + 4 = 0 \Rightarrow r = 2$ (double root)

$$y = Ae^{2t} + Bte^{2t}$$

$e^{2t} \rightsquigarrow$ **double** root $r = 2$ so

$$y_p = At^2e^{2t}$$

(h) $y'' - 2y' = t$

Aux: $r^2 - 2r = 0 \Rightarrow r = 0$ or $r = 2$

Careful, $t = te^{0t} \rightsquigarrow r = 0$ which coincides, so

$$y_p = t\,(At + B) = At^2 + Bt$$