LECTURE: INITIAL-VALUE PROBLEMS

1. Laplace Transform

Definition:

$$\mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt$$

Laplace Miracles:

$$\begin{aligned}\mathcal{L}\left\{f'(t)\right\} =& s\mathcal{L}\left\{f(t)\right\} - f(0)\\ \mathcal{L}\left\{f''(t)\right\} =& s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)\end{aligned}$$

In other words, the Laplace transform turns differentiation (hard) into multiplication (easy)

And therefore, it will turn differential equations into algebra equations:

2. Applications to ODE

Example 1:

$$\begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 4 \\ y'(0) = 9 \end{cases}$$

STEP 1: Take Laplace Transforms

$$\begin{aligned}\mathcal{L}\left\{y''-5y'+6y\right\} =&\mathcal{L}\left\{0\right\}\\ \mathcal{L}\left\{y''\right\}-5\mathcal{L}\left\{y'\right\}+6\mathcal{L}\left\{y\right\} =&0\end{aligned}$$

$$\begin{aligned}\mathcal{L}\left\{y''\right\} =&s^2\mathcal{L}\left\{y\right\}-sy(0)-y'(0)=s^2\mathcal{L}\left\{y\right\}-s(4)-9=s^2\mathcal{L}\left\{y\right\}-4s-9\\ \mathcal{L}\left\{y'\right\} =&s\mathcal{L}\left\{y\right\}-y(0)=s\mathcal{L}\left\{y\right\}-4\end{aligned}$$

$$\begin{aligned}\mathcal{L}\left\{y''\right\}-5\mathcal{L}\left\{y'\right\}+6\mathcal{L}\left\{y\right\} =&0\\ \left(s^2\mathcal{L}\left\{y\right\}-4s-9\right)-5\left(s\mathcal{L}\left\{y\right\}-4\right)+6\mathcal{L}\left\{y\right\} =&0\\ s^2\mathcal{L}\left\{y\right\}-4s-9-5s\mathcal{L}\left\{y\right\}+20+6\mathcal{L}\left\{y\right\} =&0\\ \mathcal{L}\left\{y\right\}\left(s^2-5s+6\right)-4s+11 =&0\\ \mathcal{L}\left\{y\right\} =&\frac{4s-11}{s^2-5s+6}\end{aligned}$$

Notice how this became an *algebra* equation for $\mathcal{L}\left\{y\right\}$, no differentiation whatsoever.

STEP 2: Partial Fractions

Main Idea: Recognize the right hand side as a Laplace transform.

$$\begin{aligned}\frac{4s-11}{s^2-5s+6} =&\frac{A}{s-2}+\frac{B}{s-3}=\frac{A(s-3)+B(s-2)}{(s-2)(s-3)}=\frac{As-3A+Bs-2B}{s^2-5s+6}\\ =&\frac{(A+B)s+(-3A-2B)}{s^2-5s+6}=\frac{4s-11}{s^2-5s+6}\end{aligned}$$

$$\begin{cases} A+B =4\\ -3A-2B =-11\end{cases}$$

Since $B=4-A$ we get

$$\begin{aligned} -3A - 2B &= -11 \\ -3A - 2(4 - A) &= -11 \\ -3A - 8 + 2A &= -11 \\ -A &= -11 + 8 = -3 \\ A &= 3 \end{aligned}$$

So $A = 3$ and $B = 4 - A = 4 - 3 = 1 \Rightarrow A = 3$ and $B = 1$

STEP 3: Solution

$$\mathcal{L}\{y\} = \frac{4s - 11}{s^2 - 5s + 6} = \frac{3}{s-2} + \frac{1}{s-3} = \mathcal{L}\left\{3e^{2t} + e^{3t}\right\}$$

Comparing Laplace transforms, we then get

$$y = 3e^{2t} + e^{3t}$$

Omg, why subject ourselves to this torture if we could easily just find the auxiliary equation?

The true magic will happen in a couple of lectures, where we'll use this to solve ODE where the right hand side isn't even a function any more!

Example 2:

$$\begin{cases} y'' + y = -9\sin(2t) \\ y(0) = 2 \\ y'(0) = 10 \end{cases}$$

STEP 1: Laplace Transform

$$\begin{aligned}
\mathcal{L}\{y''\} + \mathcal{L}\{y\} =& \mathcal{L}\{-9\sin(2t)\} \\
\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + \mathcal{L}\{y\} =& -9\left(\frac{2}{s^2+4}\right) \\
s^2\mathcal{L}\{y\} - 2s - 10 + \mathcal{L}\{y\} =& \frac{-18}{s^2+4} \\
\left(s^2+1\right)\mathcal{L}\{y\} =& (2s+10) - \frac{18}{s^2+4} \\
\mathcal{L}\{y\} =& \frac{2s+10}{s^2+1} - \frac{18}{\left(s^2+1\right)\left(s^2+4\right)}
\end{aligned}$$

The $\frac{2s+10}{s^2+1}$ term is already in partial fractions, so we just need to focus on the second term:

STEP 2: Partial Fractions

$$\begin{aligned}
\frac{-18}{(s^2+1)(s^2+4)} =& \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
=& \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{\left(s^2+1\right)\left(s^2+4\right)} \\
=& \frac{As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D}{\left(s^2+1\right)\left(s^2+4\right)} \\
\frac{2s^3 + 10s^2 + 8s + 22}{(s^2+1)(s^2+4)} =& \frac{(A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)}{\left(s^2+1\right)\left(s^2+4\right)}
\end{aligned}$$

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 4A + C = 0 \\ 4B + D = -18 \end{cases} \Rightarrow \begin{cases} C = -A \\ D = -B \\ 4A + (-A) = 0 \\ 4B + (-B) = -18 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = -6 \\ C = 0 \\ D = 6 \end{cases}$$

$$\frac{-18}{(s^2+1)(s^2+4)} = \frac{0s-6}{s^2+1} + \frac{0s+6}{s^2+4} = \frac{-6}{s^2+1} + \frac{6}{s^2+4}$$

STEP 3: Answer

$$\begin{aligned}
\mathcal{L}\{y\} =& \frac{2s+10}{s^2+1} - \frac{18}{(s^2+1)(s^2+4)} \\
=& \frac{2s+10}{s^2+1} - \frac{6}{s^2+1} + \frac{6}{s^2+4} \\
=& \frac{2s+4}{s^2+1} + \frac{6}{s^2+4} \\
=& 2\left(\frac{s}{s^2+1}\right) + 4\left(\frac{1}{s^2+1}\right) + 3\left(\frac{2}{s^2+4}\right) \\
=& \mathcal{L}\{2\cos(t) + 4\sin(t) + 3\sin(2t)\}
\end{aligned}$$

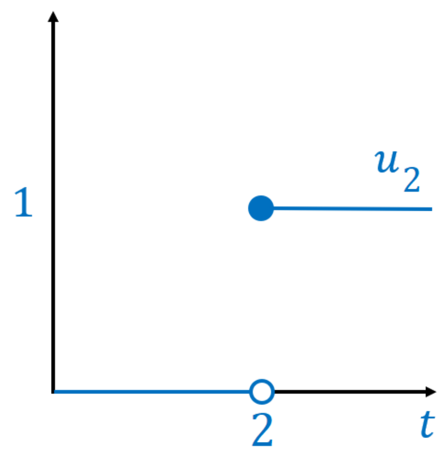
$$y = 2\cos(t) + 4\sin(t) + 3\sin(2t)$$

3. Step Functions

Our ultimate goal is to be able to solve ODE with jumps. In order to do this, we need to define:

Definition: (Heaviside Function at 2)

$$u_2(t) = \begin{cases} 0 & \text{if} \quad t < 2 \\ 1 & \text{if} \quad t \geq 2 \end{cases}$$

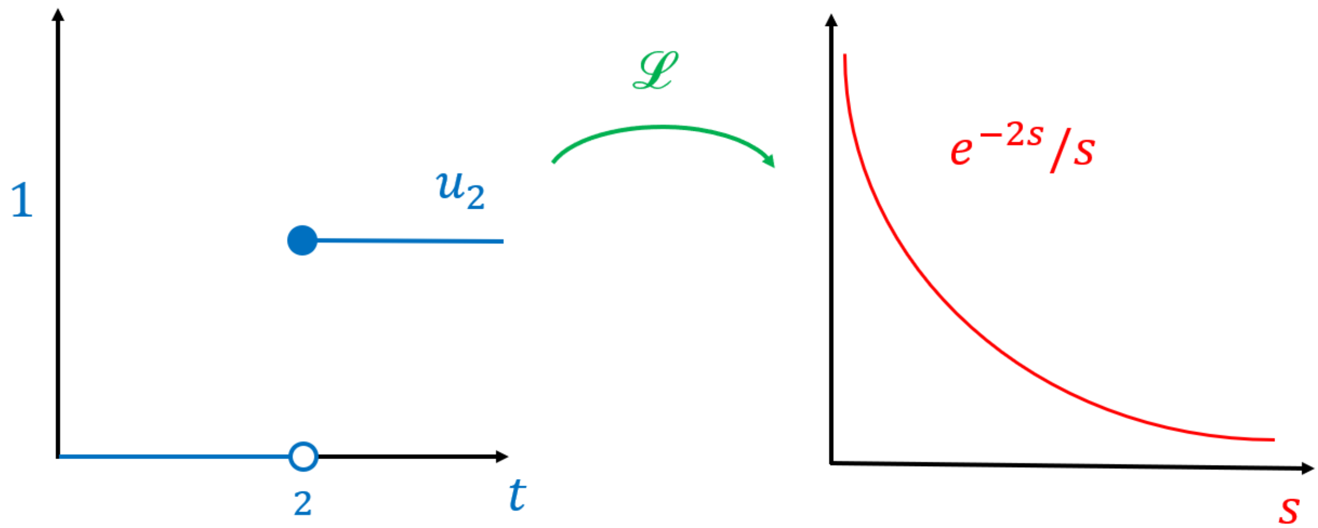


Note: u_2 called the Heaviside function at 2 and is used to represent a unit impulse (think force or voltage)

Example 3:

$$\mathcal{L}\left\{u_2(t)\right\}$$

$$\begin{aligned}
\mathcal{L}\left\{u_2(t)\right\} &= \int_0^\infty u_2(t)e^{-st}dt \\
&= \int_0^2 0e^{-st}dt + \int_2^\infty 1e^{-st}dt \\
&= \left[\frac{e^{-st}}{-s}\right]_{t=2}^{t=\infty} \\
&= \left(\lim_{t\to\infty}\frac{e^{-st}}{-s}\right) - \frac{e^{-s(2)}}{-s} \\
&= 0 + \frac{e^{-2s}}{s} = \frac{e^{-2s}}{s}
\end{aligned}$$



More generally, it follows that

Fact:

$$\mathcal{L}\left\{u_c(t)\right\} = \frac{e^{-cs}}{s}$$