LECTURE: VARIATION OF PARAMETERS

1. Variation of Parameters

Video: Variation of Parameters for Systems

Good News: The *exact* same method of variation of parameters works for systems as well!

Example 1:

Solve $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix} 7 & -3 \\ 8 & -3 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix}$$

STEP 1: Homogeneous Solution: Solve $\mathbf{x}' = A\mathbf{x}$

Eigenvalues:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 7-\lambda & -3 \\ 8 & -3-\lambda \end{vmatrix} \\ &= (7-\lambda)(-3-\lambda) - (-3)(8) \\ &= -21 - 7\lambda + 3\lambda + \lambda^2 + 24 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3) \end{aligned}$$

Which gives $\lambda = 1$ and $\lambda = 3$

$\boxed{\lambda = 1}$

$$\text{Nul }(A - 1I) = \left[\begin{array}{cc|c} 7-1 & -3 & 0 \\ 8 & -3-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 6 & -3 & 0 \\ 8 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$2x - y = 0 \Rightarrow y = 2x \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\boxed{\lambda = 3}$

$$\text{Nul }(A - 3I) = \left[\begin{array}{cc|c} 7-3 & -3 & 0 \\ 8 & -3-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 8 & -6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 4 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$4x - 3y = 0 \Rightarrow x = 3 \text{ and } y = 4$$

$$\lambda = 3 \rightsquigarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\mathbf{x_0}(t) = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = C_1 \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + C_2 \begin{bmatrix} 3e^{3t} \\ 4e^{3t} \end{bmatrix}$$

STEP 2: Variation of Parameters

$$\mathbf{x_p} = u(t) \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + v(t) \begin{bmatrix} 3e^{3t} \\ 4e^{3t} \end{bmatrix}$$

Var of Par Equations:

$$\begin{bmatrix} e^t & 3e^{3t} \\ 2e^t & 4e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix}$$

(No need to differentiate here because we have a matrix already. And the right-hand-side is the inhomogeneous term)

Denominator: $\begin{vmatrix} e^t & 3e^{3t} \\ 2e^t & 4e^{3t} \end{vmatrix} = \left(e^t\right)\left(4e^{3t}\right) - \left(3e^{3t}\right)\left(2e^t\right) = 4e^{4t} - 6e^{4t} = -2e^{4t}$

$$u'(t) = \frac{\begin{vmatrix} e^{2t} & 3e^{3t} \\ 4e^{2t} & 4e^{3t} \end{vmatrix}}{-2e^{4t}} = \frac{\left(e^{2t}\right)\left(4e^{3t}\right) - \left(3e^{3t}\right)\left(4e^{2t}\right)}{-2e^{4t}} = \frac{4e^{5t} - 12e^{5t}}{-2e^{4t}} = \frac{-8e^{5t}}{-2e^{4t}} = 4e^t$$

$$v'(t) = \frac{\begin{vmatrix} e^t & e^{2t} \\ 2e^t & 4e^{2t} \end{vmatrix}}{-2e^{4t}} = \frac{e^t\left(4e^{2t}\right) - \left(e^{2t}\right)\left(2e^t\right)}{-2e^{4t}} = \frac{4e^{3t} - 2e^{3t}}{-2e^{4t}} = \frac{2e^{3t}}{-2e^{4t}} = -e^{-t}$$

$$u(t) = \int 4e^t dt = 4e^t$$

$$v(t) = \int -e^{-t} dt = e^{-t}$$

$$\mathbf{x_p}(t) = u(t)\begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + v(t)\begin{bmatrix} 3e^{3t} \\ 4e^{3t} \end{bmatrix} = 4e^t\begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + e^{-t}\begin{bmatrix} 3e^{3t} \\ 4e^{3t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} + 3e^{2t} \\ 8e^{2t} + 4e^{2t} \end{bmatrix} = \begin{bmatrix} 7e^{2t} \\ 12e^{2t} \end{bmatrix}$$

STEP 3: General Solution

$$\mathbf{x}(t) = \mathbf{x_0} + \mathbf{x_p} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} 3 \\ 4 \end{bmatrix} + e^{2t}\begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

2. Why this works

In terms of the example above, let

$$\mathbf{x_1}(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix} \qquad \mathbf{x_2}(t) = \begin{bmatrix} 3e^{3t} \\ 4e^{3t} \end{bmatrix}$$

Note that $\mathbf{x_1}$ and $\mathbf{x_2}$ solve the homogeneous equation $\mathbf{x}' = A\mathbf{x}$

$$\text{Suppose } \mathbf{x_p} = u(t)\mathbf{x_1}(t) + v(t)\mathbf{x_2}(t)$$

Plug this into the system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$

$$\begin{aligned} \mathbf{x_p}' &= A\mathbf{x_p} + \mathbf{f} \\ \left(u(t)\mathbf{x_1}(t) + v(t)\mathbf{x_2}(t)\right)' &= A\left(u(t)\mathbf{x_1}(\mathbf{t}) + v(t)\mathbf{x_2}(t)\right) + \mathbf{f} \\ u'(t)\mathbf{x_1} + \cancel{u(t)\mathbf{x_1}'} + v'(t)\mathbf{x_2} + \cancel{v(t)\mathbf{x_2}'} &= \cancel{u(t)A\mathbf{x_1}} + \cancel{v(t)A\mathbf{x_2}} + \mathbf{f} \end{aligned}$$

The terms cancel out because $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, so $\mathbf{x_1}' = A\mathbf{x_1}$ and $\mathbf{x_2}' = A\mathbf{x_2}$

Therefore we're left with

$$u'(t)\mathbf{x_1} + v'(t)\mathbf{x_2} = \mathbf{f}$$

$$u'(t)\begin{bmatrix} e^t \\ 2e^t \end{bmatrix} + v'(t)\begin{bmatrix} 3e^{3t} \\ 4e^{3t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e^t & 3e^{3t} \\ 2e^t & 4e^{3t} \end{bmatrix}\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix}$$

Which are the Var of Par equations □

Aside: If you want to see a more direct method of solving inhomogeneous systems, check out the video below

Video: Solving systems using Laplace transforms

3. Trig Example

Example 2:

Find a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} -\cos(t) \\ 5\sin(t) \end{bmatrix}$$

STEP 1: Homogeneous Solution: Solve $\mathbf{x}' = A\mathbf{x}$

Eigenvalues:

$$\begin{aligned} |A - \lambda I| = & \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} \\ = & (2-\lambda)(-2-\lambda) - (-5)(1) \\ = & -4 - \cancel{2\lambda} + \cancel{2\lambda} + \lambda^2 + 5 \\ = & \lambda^2 + 1 = 0 \\ \Rightarrow & \lambda = \pm i \end{aligned}$$

$$\text{Nul } (A - iI) = \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x - (2+i)y = 0 \Rightarrow x = (2+i)y$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (2+i)y \\ y \end{bmatrix} = y \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

$$\lambda = i \rightsquigarrow \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

$$e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t)) \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\begin{aligned}\mathbf{x_0}(t) =& C_1 \left(\cos(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + C_2 \left(\cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \\ =& C_1 \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}\end{aligned}$$

STEP 2: Variation of Parameters

$$\mathbf{x_p}(t) = u(t) \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + v(t) \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}$$

$$\begin{bmatrix} 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -\cos(t) \\ 5\sin(t) \end{bmatrix}$$

Denominator:

$$\begin{aligned}& \begin{vmatrix} 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \\ \cos(t) & \sin(t) \end{vmatrix} \\ =& \sin(t)\left(2\cos(t) - \sin(t)\right) - \cos(t)\left(\cos(t) + 2\sin(t)\right) \\ =& \cancel{2\sin(t)\cos(t)} - \sin^2(t) - \cos^2(t) - \cancel{2\cos(t)\sin(t)} \\ =& -1\end{aligned}$$

$$\begin{aligned}u'(t) =& \frac{\begin{vmatrix} -\cos(t) & \cos(t) + 2\sin(t) \\ 5\sin(t) & \sin(t) \end{vmatrix}}{-1} \\ =& -\left[-\cos(t)\sin(t) - 5\sin(t)\left(\cos(t) + 2\sin(t)\right)\right] \\ =& \cos(t)\sin(t) + 5\sin(t)\cos(t) + 10\sin^2(t) \\ =& 6\cos(t)\sin(t) + 10\sin^2(t)\end{aligned}$$

$$\begin{aligned}
v'(t) &= -\begin{vmatrix} 2\cos(t) - \sin(t) & -\cos(t) \\ \cos(t) & 5\sin(t) \end{vmatrix} \\
&= -\left[10\cos(t)\sin(t) - 5\sin^2(t) + \cos^2(t)\right] \\
&= -10\cos(t)\sin(t) + 5\sin^2(t) - \cos^2(t)
\end{aligned}$$

$$\begin{aligned}
u(t) &= \int 6\cos(t)\sin(t) + 10\sin^2(t)dt \\
&= \int 6\left(\frac{1}{2}\sin(2t)\right) + 10\left(\frac{1-\cos(2t)}{2}\right)dt \\
&= \int 3\sin(2t) + 5 - 5\cos(2t)dt \\
&=3\left(\frac{-\cos(2t)}{2}\right) + 5t - 5\left(\frac{\sin(2t)}{2}\right) \qquad \text{Use } u = 2t \\
&= -\frac{3}{2}\cos(2t) + 5t - \frac{5}{2}\sin(2t)
\end{aligned}$$

$$\begin{aligned}
v(t) &= \int -10\cos(t)\sin(t) + 5\sin^2(t) - \cos^2(t)dt \\
&= \int -10\left(\frac{1}{2}\sin(2t)\right) + 5\left(\frac{1-\cos(2t)}{2}\right) - \left(\frac{1+\cos(2t)}{2}\right)dt \\
&= \int -5\sin(2t) + \frac{5}{2} - \frac{5}{2}\cos(2t) - \frac{1}{2} - \frac{1}{2}\cos(2t)dt \\
&= \int -5\sin(2t) + 2 - 3\cos(2t)dt \\
&=\frac{5}{2}\cos(2t) + 2t - \frac{3}{2}\sin(2t) \qquad \text{Use } u = 2t
\end{aligned}$$

$$\begin{aligned}\mathbf{x_p}(t) =& u(t)\begin{bmatrix}2\cos(t)-\sin(t)\\ \cos(t)\end{bmatrix}+v(t)\begin{bmatrix}\cos(t)+2\sin(t)\\ \sin(t)\end{bmatrix}\\ =& \left(-\frac{3}{2}\cos(2t)+5t-\frac{5}{2}\sin(2t)\right)\begin{bmatrix}2\cos(t)-\sin(t)\\ \cos(t)\end{bmatrix}\\ &+\left(\frac{5}{2}\cos(2t)+2t-\frac{3}{2}\sin(2t)\right)\begin{bmatrix}\cos(t)+2\sin(t)\\ \sin(t)\end{bmatrix}\end{aligned}$$

4. More Practice

Example 3: (more practice)

Solve $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix}2e^t\\ 4t\end{bmatrix}$$

STEP 1: Homogeneous Solution: Solve $\mathbf{x}' = A\mathbf{x}$

Eigenvalues:

$$\begin{aligned}|A-\lambda I| =& \begin{vmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{vmatrix}\\ =& (2-\lambda)^2-(-1)(-1)\\ =& (\lambda-2)^2-1=0\end{aligned}$$

$$\lambda-2=\pm 1 \Rightarrow \lambda = 1 \text{ or } 3$$

$\boxed{\lambda = 1}$

$$\text{Nul }(A-1I) = \left[\begin{array}{cc|c}2-1 & -1 & 0\\ -1 & 2-1 & 0\end{array}\right] = \left[\begin{array}{cc|c}1 & -1 & 0\\ -1 & 1 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$x - y = 0 \Rightarrow x = y$$

$$\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\boxed{\lambda = 3}$

$$\text{Nul } (A - 3I) = \left[\begin{array}{cc|c} 2-3 & -1 & 0 \\ -1 & 2-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + y = 0 \Rightarrow y = -x$$

$$\lambda = 3 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x_0}(t) = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = C_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix}$$

STEP 2: Variation of Parameters

$$\mathbf{x_p} = u(t) \begin{bmatrix} e^t \\ e^t \end{bmatrix} + v(t) \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} e^t & e^{3t} \\ e^t & -e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 2e^t \\ 4t \end{bmatrix}$$

$$\text{Denominator: } \begin{vmatrix} e^t & e^{3t} \\ e^t & -e^{3t} \end{vmatrix} = e^t\left(-e^{3t}\right) - e^t e^{3t} = -e^{4t} - e^{4t} = -2e^{4t}$$

$$u'(t) = \frac{\begin{vmatrix} 2e^t & e^{3t} \\ 4t & -e^{3t} \end{vmatrix}}{-2e^{4t}} = \frac{-2e^{4t} - 4te^{3t}}{-2e^{4t}} = 1 + 2te^{-t}$$

$$v'(t) = \frac{\begin{vmatrix} e^t & 2e^t \\ e^t & 4t \end{vmatrix}}{-2e^{4t}} = \frac{4te^t - 2e^{2t}}{-2e^{4t}} = -2te^{-3t} + e^{-2t}$$

$$\begin{aligned}
u(t) &= \int 1 + 2te^{-t} dt \\
&\overset{\text{IBP}}{=} t + 2t\left(-e^{-t}\right) - \int 2\left(-e^{-t}\right) dt \qquad u = 2t,\ dv = e^{-t} \\
&= t - 2te^{-t} + 2\int e^{-t} dt \\
&= t - 2te^{-t} - 2e^{-t}
\end{aligned}$$

$$\begin{aligned}
v(t) &= \int -2te^{-3t} + e^{-2t} dt \\
&\overset{\text{IBP}}{=} -2t\left(\frac{e^{-3t}}{-3}\right) - \int -2\left(\frac{e^{-3t}}{-3}\right) dt - \frac{1}{2}e^{-2t} \qquad u = -2t,\ dv = e^{-3t} \\
&= \frac{2}{3}te^{-3t} - \frac{2}{3}\int e^{-3t} dt - \frac{1}{2}e^{-2t} \\
&= \frac{2}{3}te^{-3t} - \frac{2}{3}\left(\frac{e^{-3t}}{-3}\right) - \frac{1}{2}e^{-2t} \\
&= \frac{2}{3}te^{-3t} + \frac{2}{9}e^{-3t} - \frac{1}{2}e^{-2t}
\end{aligned}$$

$$\begin{aligned}
\mathbf{x_p}(t) &= u(t)\begin{bmatrix} e^t \\ e^t \end{bmatrix} + v(t)\begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} \\
&= \left(t - 2te^{-t} - 2e^{-t}\right)\begin{bmatrix} e^t \\ e^t \end{bmatrix} + \left(\frac{2}{3}te^{-3t} + \frac{2}{9}e^{-3t} - \frac{1}{2}e^{-2t}\right)\begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} \\
&= \left(te^t - 2t - 2\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{2}{3}t + \frac{2}{9} - \frac{1}{2}e^t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

STEP 3: General Solution

$$\begin{aligned}\mathbf{x}(t) =& \mathbf{x_0}(t) + \mathbf{x_p}(t) \\ =& C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left(te^t - 2t - 2\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{2}{3}t + \frac{2}{9} - \frac{1}{2}e^t\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

5. Nonlinear Systems

Welcome to the magical world of *nonlinear* systems! They are more complicated but also much more interesting, and we will see many cool real-life applications.

Example 4:

$$\begin{cases} x'(t) = x^2 + xy + e^y \\ y'(t) = \cos(x) + 2y \end{cases}$$

Note: We can write this in the form $\mathbf{x}'(t) = F(\mathbf{x}(t))$ where

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \text{ and } F(x, y) = \begin{bmatrix} x^2 + xy + e^y \\ \cos(x) + 2y \end{bmatrix}$$

Question: Does this system even have a solution?

Recall: Existence-Uniqueness Theorem

$y' = f(y, t)$ has a unique solution if f and $\frac{\partial f}{\partial y}$ are continuous.

Same thing here, except we differentiate F with respect to everything

Definition:

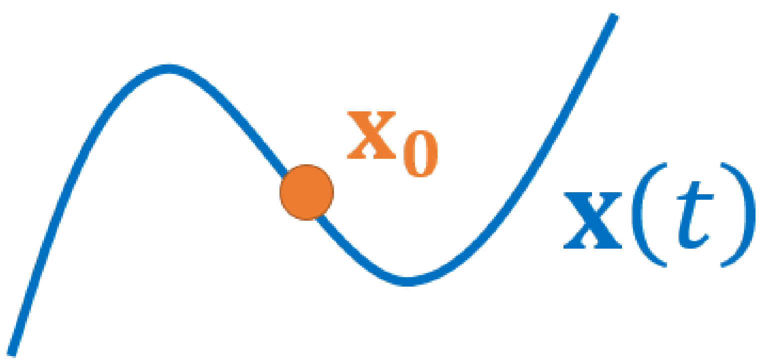
$$\nabla F = \begin{bmatrix} \frac{\partial\left(x^2+xy+e^y\right)}{\partial x} & \frac{\partial\left(x^2+xy+e^y\right)}{\partial y} \\ \frac{\partial(\cos(x)+2y)}{\partial x} & \frac{\partial(\cos(x)+2y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x+y & x+e^y \\ -\sin(x) & 2 \end{bmatrix}$$

Theorem:

If F and ∇F are continuous, then the ODE

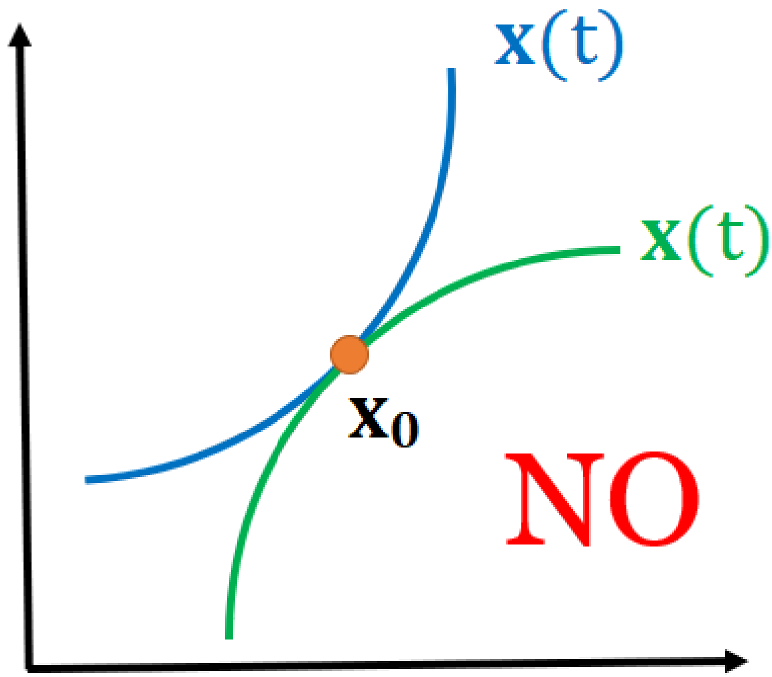
$$\begin{cases} \mathbf{x}' = F(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x_0} \end{cases}$$

Has a unique solution $\mathbf{x}(t)$ for t close enough to 0



So here the answer here is **YES**

Note: This implies again that solutions/phase portraits cannot cross, otherwise we would violate uniqueness.



6. Equilibrium Solutions

Recall:

To find equilibrium solutions of $y' = 2y\left(1 - \frac{y}{3}\right)$ we set $y' = 0$

Example 5:

Find the equilibrium solutions of

$$\begin{cases} x' = -x + xy \\ y' = -8y + 4xy \end{cases}$$

Here you set $x' = 0$ **and** $y' = 0$

$$\begin{cases} -x + xy &= 0 \\ -8y + 4xy &= 0 \end{cases} \Rightarrow \begin{cases} x(-1+y) &= 0 \\ y(-8+4x) &= 0 \end{cases} \Rightarrow \begin{cases} x &= 0 \text{ or } y = 1 \textbf{ AND} \\ y &= 0 \text{ or } x = 2 \end{cases}$$

Analogy: Think of $x = 0$ or $y = 1$ as Food and $y = 0$ or $x = 2$ as Drinks. You first choose one food and then one drink.

Case 1: If $x = 0$ then either $y = 0$ or ~~$x = 2$~~ which gives $(0, 0)$

Case 2: If $y = 1$ then either ~~$y = 0$~~ or $x = 2$ which gives $(2, 1)$

Answer: $(0, 0)$ and $(2, 1)$

Interpretation: Solutions starting at $(0, 0)$ stay at $(0, 0)$ and similarly with $(2, 1)$

7. Classification of equilibrium points

Recall:

We classified equilibrium solutions like $y = 0$ and $y = 3$ as stable/unstable/bistable

Example 6:

Classify the equilibrium points $(0, 0)$ and $(2, 1)$

Case 1: $(0, 0)$

STEP 1: Find $\nabla F(0, 0)$

$$\nabla F(x, y) = \begin{bmatrix} \frac{\partial(-x+xy)}{\partial x} & \frac{\partial(-x+xy)}{\partial y} \\ \frac{\partial(-8y+4xy)}{\partial x} & \frac{\partial(-8y+4xy)}{\partial y} \end{bmatrix} = \begin{bmatrix} -1+y & x \\ 4y & -8+4x \end{bmatrix}$$

$$\nabla F(0,0) = \begin{bmatrix} -1+0 & 0 \\ 4(0) & -8+4(0) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = A$$

STEP 2: The eigenvalues of $A = \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix}$ are $\lambda = -1 < 0$ and $\lambda = -8 < 0$

Hence $(0,0)$ is **stable**

(To be continued next time)