APMA 0350 – MIDTERM 1 – SOLUTIONS

1. **Separation of variables**

STEP 1:

$$\begin{aligned}
\frac{dy}{dt} =& te^{2y} \\
e^{-2y}\,dy =& t\,dt \\
\int e^{-2y} dy =& \int t dt \\
-\frac{1}{2}e^{-2y} =& \frac{1}{2}t^2 + C \\
e^{-2y} =& -t^2 \underbrace{-2C}_{C} = -t^2 + C \\
-2y =& \ln\left(-t^2 + C\right) \\
y =& -\frac{1}{2}\ln\left(-t^2 + C\right)
\end{aligned}$$

STEP 2:

$$\begin{aligned}
y(0) =& -1 \\
-\frac{1}{2}\ln\left(-0^2 + C\right) =& -1 \\
\ln(C) =& 2 \\
C =& e^2
\end{aligned}$$

STEP 3: Solution

$$y = -\frac{1}{2}\ln\left(-t^2 + e^2\right)$$

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2. Integrating Factors

STEP 1: Standard Form

$$y' + \left(\frac{\sin(t)}{\cos(t)}\right) y = \sin(t)$$

STEP 2: Integrating Factor

$$e^{\int \frac{\sin(t)}{\cos(t)} dt} = e^{\int \tan(t) dt} = e^{\ln|\sec(t)| dt} = |\sec(t)| = \sec(t)$$

In the last step, we used $-\frac{\pi}{2} < t < \frac{\pi}{2}$

STEP 3: Multiply by $\sec(t)$

$$\sec(t) y' + \sec(t) \left(\frac{\sin(t)}{\cos(t)}\right) y = \sec(t) \sin(t)$$

$$\begin{aligned}
(\sec(t) y)' &= \sec(t)\sin(t) = \left(\frac{1}{\cos(t)}\right) \sin(t) = \tan(t) \\
\sec(t) y &= \int \tan(t) dt = \ln(\sec(t)) + C \\
y &= \frac{\ln(\sec(t)) + C}{\sec(t)} \\
y &= \cos(t) \left(\ln(\sec(t)) + C\right)
\end{aligned}$$

STEP 4: Initial Condition

$$\begin{aligned}
y(0) &= 3 \\
\cos(0) \left(\ln(\sec(0)) + C\right) &= 3 \\
\ln(1) + C &= 3 \\
C &= 3
\end{aligned}$$

STEP 5: Solution

$$y = \cos(t) \left(\ln(\sec(t)) + 3\right)$$

3. **Exact Equations**

STEP 1: Check Exact

$$P_y = \left(6y^3x + \cos(y)\right)_y = 6\left(3y^2\right)x - \sin(y) = 18xy^2 - \sin(y)$$
$$Q_x = \left(k\,x^2y^2 - x\sin(y)\right)_x = k(2x)y^2 - \sin(y) = 2kxy^2 - \sin(y)$$

The equation is exact iff $P_y = Q_x$ and so $18 = 2k$ that is $\boxed{k = 9}$

STEP 2:

$$f_x = 6y^3x + \cos(y) \Rightarrow f = \int 6xy^3 + \cos(y)dx = 3x^2y^3 + x\cos(y) + g(y)$$
$$f_y = 9x^2y^2 - x\sin(y) \Rightarrow f = \int 9x^2y^2 - x\sin(y)dy = 3x^2y^3 + x\cos(y) + h(x)$$

Therefore $f(x,y) = 3x^2y^3 + x\cos(y)$

STEP 3: Answer:

$$3x^2y^3 + x\cos(y) = C$$

4. **STEP 1:** Differential Equation

From the assumptions of the problem, we get

$$T' = k\,(T + T_0)$$

Here $T_0 = 1$ is the ambient temperature, and therefore

$$T' = k\,(T + 1)$$

STEP 2: Integrating Factors

$$\begin{aligned} T' &= k\,(T+1) = kT + k \\ T' - kT &= k \end{aligned}$$

Multiply by e^{-kt} to get

$$\begin{aligned} e^{-kt}T' - ke^{-kt}T &= ke^{-kt} \\ \left(e^{-kt}T\right)' &= ke^{-kt} \\ e^{-kt}T &= \int ke^{-kt}dt \\ e^{-kt}T &= -e^{-kt} + C \\ T &= -1 + Ce^{kt} \end{aligned}$$

STEP 3: Solve for C and k

$$\begin{aligned} T(0) &= 5 \\ -1 + Ce^0 &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

$$T = -1 + 6e^{kt}$$

$$\begin{aligned} T(1) &= 11 \\ -1 + 6e^k &= 11 \\ 6e^k &= 12 \\ e^k &= 2 \\ k &= \ln(2) \end{aligned}$$

$$T = -1 + 6e^{\ln(2)t} = -1 + 6\left(e^{\ln(2)}\right)^t = -1 + 6\left(2^t\right)$$

STEP 3:

$$\begin{aligned} T &= 23 \\ -1 + 6\left(2^t\right) &= 23 \\ 6\left(2^t\right) &= 24 \\ 2^t &= 4 \\ t &= 2 \end{aligned}$$

Therefore the temperature reaches 23° C after $\boxed{t = 2 \text{ minutes}}$

5. **STEP 1: Auxiliary Equation**

$r^2 - 7r + 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow r = 2$ or $r = 5$

$$y = Ae^{2t} + Be^{5t}$$

STEP 2: Initial Condition

$y(0) = 6 \Rightarrow Ae^0 + Be^0 = 6 \Rightarrow A + B = 6 \Rightarrow B = 6 - A$

$$y = Ae^{2t} + (6-A)\,e^{5t}$$
$$y' = 2Ae^{2t} + 5(6-A)e^{5t}$$

$$y'(0) = 2Ae^0 + 5(6-A)e^0 = 2A + 30 - 5A = -3A + 30 = 9$$

This gives $-3A = -21$ so $A = 7$

And $B = 6 - A = 6 - 7 = -1$

STEP 3: Answer

$$y = 7e^{2t} - e^{5t}$$